Last Time

- We computed

$$G_R(t\vec{r}\,|\,t_0\vec{r}_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\,\delta\!\left(t-t_0-\frac{|\vec{r}-\vec{r}_0|}{c}\right)$$

The green fcn of the wave equation

- Of course we have a definite reason why.

Starting from Maxwell

$$\nabla\cdot E = \rho$$

$$\nabla\times B = \frac{j}{c} + \frac{1}{c}\partial_t E$$

$$\nabla\cdot B = 0$$

$$\nabla\times\vec{E} = -\frac{1}{c}\partial_t\vec{B}$$

We introduce φ and $\vec{A}$ $\quad B = \nabla\times\vec{A} \quad \vec{E} = -\frac{1}{c}\partial_t\vec{A} - \nabla\varphi$
and found

$$-\Box\varphi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot\vec{A}\right) = \rho$$

$$-\Box\vec{A} + \vec{\nabla}\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot\vec{A}\right) = \vec{J}/c$$

Selecting the Lorentz Gauge $\frac{1}{c}\partial_t \varphi + \nabla \cdot \vec{A} = 0$

We have

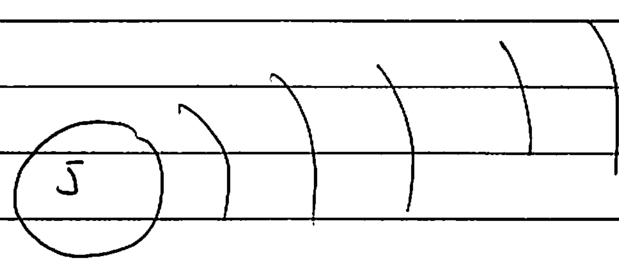
$$\boxed{\begin{aligned} -\Box \varphi &= \rho \qquad\qquad \Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \\ -\Box \vec{A} &= \vec{j}/c \end{aligned}}$$

Solving For the Waves:

$$\varphi(t,r) = \int d^3r_0 \, dt_0' \; G(tr|t_0 r_0)\, \rho(t_0 r_0)$$

$$A(t,r) = \int d^3r_0 \, dt_0 \; G(tr|t_0 r_0)\, \frac{\vec{j}}{c}(t_0, r_0)$$

We will be initially interested in fields far from the source



Explaiting the δ-fcn we have

$$\delta\left(t - t_0 - \frac{|\vec{r}-\vec{r}_0|}{c}\right) \Rightarrow t_0 = t - \frac{|\vec{r}-\vec{r}_0|}{c}$$

Solving For waves pg. 2

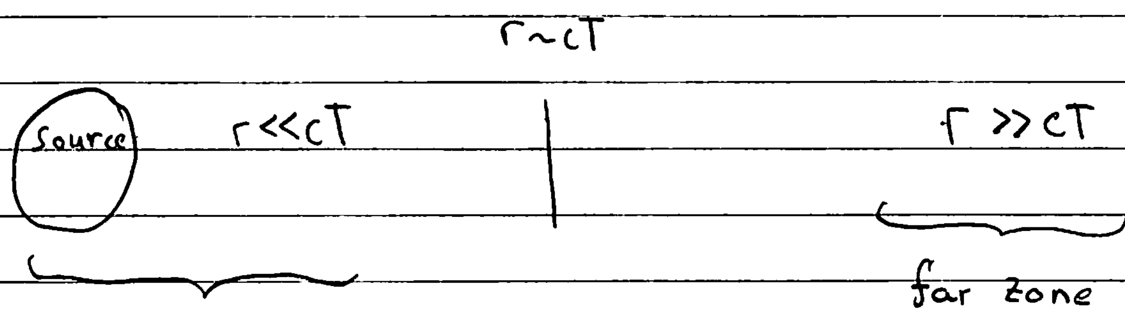
Using the Grn-fcn we have:

$$\vec{A}(t,r) = \int \frac{d^3 r_0}{4\pi |\vec{r} - r_0|} \frac{\vec{j}}{c}\Big(t - \overbrace{\frac{|\vec{r} - \vec{r}_0|}{c}}^{\text{retarded time}}, \vec{r}_0\Big)$$

$$\varphi(t,r) = \int \frac{d^3 r}{4\pi |\vec{r} - r_0|} \rho\Big(t - \frac{|\vec{r} - \vec{r}_0|}{c}, \vec{r}_0\Big)$$

Picture

- Source has size L and time scale T, will eventually simplify taking $L/T \ll c$. Then what you see depends on how far you are from the source (r = distance from source).



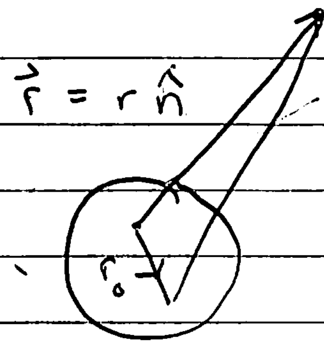
This range we already studied. This is the quasi-static region. Changes in source are "instantly" communicated to the fields. Go slightly beyond static approx by including time derivs order by order

far zone or radiation zone. Only fields which decrease as $\frac{1}{r}$, "Radiation fields", are relevant.

Long Distance Expansion

Now at great distances we can expand

$$|\vec{r} - \vec{r}_0| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}_0 + \vec{r}_0} \simeq r - \hat{n}\cdot\vec{r}_0 \qquad \hat{n} \equiv \hat{r}$$



much smaller than $\frac{r}{c}$ for non.-rell sytems

$$\text{So} \quad t - \frac{|\vec{r} - \vec{r}_0|}{c} \simeq t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c} \equiv T$$

← retarded time

So find up to $O(1/r^2)$:

$$\vec{A}(t, \vec{r}) = \frac{1}{4\pi r}\int d^3r_0 \, \frac{\vec{j}}{c}(T, \vec{r}_0)$$

$$\varphi(t, r) = \frac{1}{4\pi r}\int d^3r_0 \, \rho(T, \vec{r}_0)$$

Then we want to find the electric and magnetic fields:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$$

Computing the Fields Pg. 1

- First note, we have switched variables from $t, r_0 \longrightarrow T$ and r_0

$$T = t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c}$$

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial t}$$

$$\left(\frac{\partial}{\partial \vec{r}_0}\right)_T = \left(\frac{\partial}{\partial \vec{r}_0}\right)_t - \frac{\vec{n}}{c}\frac{\partial}{\partial t}$$

- Now take derivatives to determine $\vec{E}$ and $\vec{B}$. This is easy in the far zone:

  - $$\frac{\partial}{\partial r^i}\frac{1}{r} = \frac{-1}{r^2} n_i$$ ← suppressed by extra $\frac{1}{r}$ so can ignore

  - $$\frac{\partial}{\partial r^i}\vec{J}(T) = +\frac{\partial \vec{J}}{\partial T}\frac{\partial T}{\partial r^i}$$

$$= -\frac{\partial \vec{J}}{\partial t}\frac{n_i}{c}$$

$$T = t - \frac{r}{c} + \frac{\hat{n}\cdot r_0}{c}$$

$$\frac{\partial T}{\partial r^i} = -\frac{n_i}{c} + O\left(\frac{1}{r}\right)$$

note

$$\frac{\partial \vec{n}}{\partial r^i} = O\left(\frac{1}{r}\right)$$

Computing the fields pg. 2

So with these steps

$$\boxed{\vec{B} = -\frac{\vec{n}/c}{4\pi r} \times \int_{r_0} \frac{1}{c}\frac{\partial \vec{J}(T, r_0)}{\partial t}}$$

Similarly

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla_r \varphi \qquad -\nabla_r \rho(T, r_0) = -\frac{\partial \rho}{\partial T}\nabla_r T$$

$$= -\frac{1}{c}\frac{1}{4\pi r}\int_{r_0} \frac{1}{c}\partial_t \vec{J}(T) + \frac{\vec{n}}{c}\frac{1}{4\pi r}\int_{r_0} \frac{\partial \rho(T, r_0)}{\partial T}$$

Now use continuity:

$$\frac{\partial \rho}{\partial T} = -\left(\nabla_{r_0}\cdot \vec{J}\right)_T$$

See previous page

$$= -\left(\nabla_{r_0}\cdot\vec{J}\right)_t + \frac{\vec{n}}{c}\cdot\frac{\partial \vec{J}}{\partial t}$$

Yielding

$$\vec{E} = \frac{1}{4\pi r\cdot c^2}\int_{r_0} \underbrace{-\partial_t \vec{J}(T) + \vec{n}\left(\vec{n}\cdot\partial_t J(T)\right)}_{\text{transverse current}}$$

using for any transverse vector

$$\vec{V}_T \equiv V - \vec{n}(\vec{n}\cdot\vec{V}) = -\vec{n}\times(\vec{n}\times\vec{V})$$

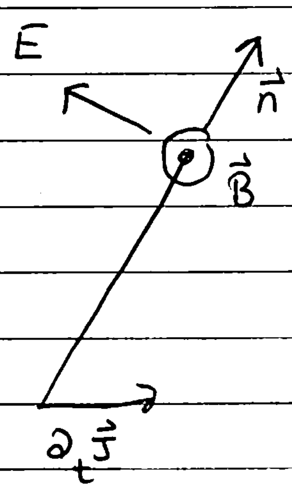
Yields

$$\boxed{\vec{E} = \vec{n}\times\left[\frac{\vec{n}}{c}\times\frac{1}{4\pi r}\int_{r_0}\frac{1}{c}\frac{\partial\vec{J}(T)}{\partial t}\right]} = -$$

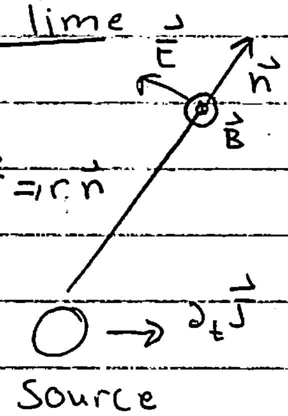
$$\boxed{\vec{E} = -\vec{n}\times\vec{B}}$$

Summary & Last Time

- Thus we see that the outgoing $\vec{E}$ & $\vec{B}$ fields are transversely polarized



Last Time



- Last time we worked out the far field behavior by solving the Wave eqn:

$$\vec{A}_{rad}(t,\vec{r}) = \frac{1}{4\pi r}\int_{r_0} \frac{\vec{J}}{c}(T,\vec{r}_0)$$

with

$$T = t - \frac{|\vec{r}-\vec{r}_0|}{c} \simeq t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c}$$

Then found:

$$\vec{B} = -\frac{\vec{n}}{c}\times\frac{\partial \vec{A}_{rad}}{\partial t}$$

and

$$\vec{E} = -\vec{n}\times\vec{B}$$

The confusion came from the change of vars $t, r_0 \to T, r_0$:

$$\left(\frac{\partial}{\partial T}\right)_{r_0} = \left(\frac{\partial}{\partial t}\right)_{r_0} \quad \text{but} \quad \left(\frac{\partial}{\partial \vec{r}_0}\right)_T = \left(\frac{\partial}{\partial \vec{r}_0}\right)_t - \frac{\vec{n}}{c}\frac{\partial}{\partial t}$$

Summary and Last Time pg. 2

Found:

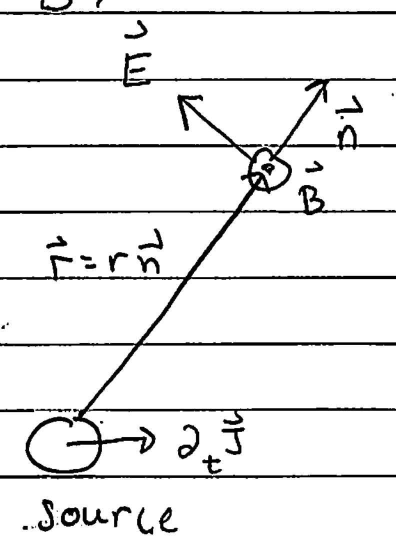
$$\vec{A}_{rad}(t,r) = \frac{1}{4\pi r} \int_{r_0} \frac{1}{c} \vec{J}(T, r_0)$$

Then

$$\vec{B} = -\frac{\vec{n}}{c} \times \frac{\partial \vec{A}_{rad}(t,\vec{r})}{\partial t}$$

$$\vec{E} = -\vec{n} \times \vec{B}(t,\vec{r})$$

The Power is:



$$dP = \vec{S} \cdot d\vec{a} \qquad \vec{S} = c\,\vec{E}\times\vec{B}$$

$$dP = r^2\, \vec{S}\cdot\vec{n}\, d\Omega$$

$$dP = c\, r^2 |\vec{E}|^2\, d\Omega$$

$$\boxed{\frac{dP}{d\Omega} = c\, r^2 |\vec{E}|^2}$$

energy per time per Ω