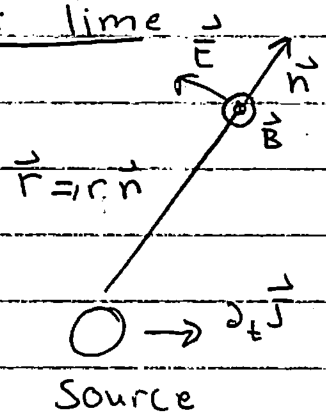
Last Time



- Last time we worked out the far field behavior by solving the Wave eqn:

$$\vec{A}_{rad}(t, \vec{r}) = \frac{1}{4\pi r} \int_{r_0} \frac{\vec{J}}{c}(T, \vec{r}_0)$$

with

$$T = t - \frac{|\vec{r} - \vec{r}_0|}{c} \simeq t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c}$$

Then found:

$$\vec{B} = -\frac{\vec{n}}{c} \times \frac{\partial \vec{A}_{rad}}{\partial t}$$

and

$$\vec{E} = -\vec{n} \times \vec{B}$$

The confusion came from the change of vars $t, r_0 \to T, r_0$:

$$\left(\frac{\partial}{\partial T}\right)_{r_0} = \left(\frac{\partial}{\partial t}\right)_{r_0} \quad \text{but} \quad \left(\frac{\partial}{\partial \vec{r}_0}\right)_T = \left(\frac{\partial}{\partial \vec{r}_0}\right)_t - \frac{\vec{n}}{c}\frac{\partial}{\partial t}$$

Summary and Last Time pg. 2

Found:

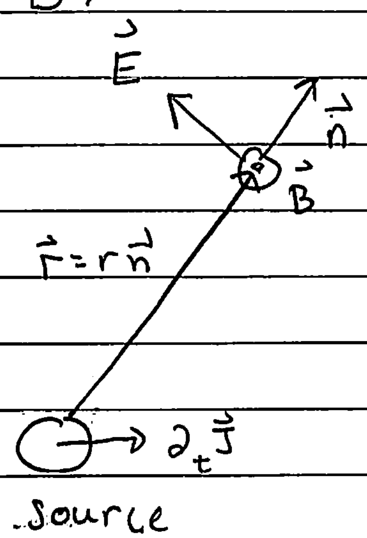
$$\vec{A}_{rad}(t,\vec{r}) = \frac{1}{4\pi r}\int_{r_0} \frac{1}{c}\,\vec{J}(T,r_0)$$

Then

$$\vec{B} = -\frac{\vec{n}}{c}\times \frac{\partial \vec{A}_{rad}(t,\vec{r})}{\partial t}$$

$$\vec{E} = -\vec{n}\times\vec{B}(t,\vec{r})$$

The Power is:



$$dP = \vec{S}\cdot d\vec{a} \qquad \vec{S} = c\,\vec{E}\times\vec{B}$$

$$dP = r^2\,\vec{S}\cdot\vec{n}\,d\Omega$$

$$dP = c\,r^2|\vec{E}|^2\,d\Omega$$

$$\boxed{\frac{dP}{d\Omega} = c\,r^2|\vec{E}|^2}$$

energy per time per Ω

Summary and Last Time Pg. 3

Can also be written in terms of $\vec{A}_{rad}$

$$\frac{dP}{d\Omega} = c \left| \vec{n} \times \underbrace{\frac{r}{c}\frac{\partial \vec{A}_{rad}}{\partial t}} \right|^2$$

factor of r strips off leading $1/r$

Larmor Formula

- Consider an accelerating charge moving non-relativistically. Calculate the power emitted.

$$\vec{J}(t_0, r_0) = e\,\vec{v}(t_0)\,\delta^3(\vec{r}_0 - \vec{R}(t_0))$$

position of particle

Then

$$\vec{A}(t,r) = \frac{1}{4\pi r}\int_{r_0} \frac{1}{c}\vec{J}(T) \qquad T = t - \frac{r}{c} - \frac{\vec{n}\cdot\vec{r}_0}{c}$$

$$= \frac{1}{4\pi r}\int_{r_0} \frac{1}{c} J\left(t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c}\right)$$

Now approximate. Great distances $r \sim ct$, but slow $\frac{r_0}{c} \ll t$

$$T \equiv t - \frac{r}{c} - \frac{\vec{n}\cdot\vec{r}_0}{c} \simeq t - \frac{r}{c} \equiv t_e \leftarrow \text{emission time}$$

So

$$\vec{A} = \frac{1}{4\pi r}\int_{r_0} e\,\frac{v(t-\frac{r}{c})}{c}\,\delta^3(\vec{r}_0 - \vec{R}(t-\tfrac{r}{c}))$$

$$\vec{A} = \frac{1}{4\pi r}\, e\,\frac{v(t-r/c)}{c}$$

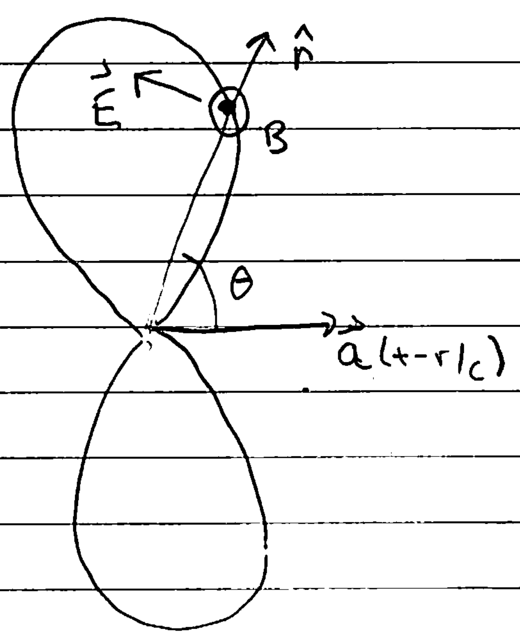
$$\vec{B} = \frac{\vec{n}}{c}\times\frac{\partial\vec{A}}{\partial t} = \frac{e}{4\pi r}\,\vec{n}\times\frac{\vec{a}(t-r/c)}{c^2} \propto a\sin\theta$$

Then power is the square of this:

$$\frac{dP}{d\Omega} = c\cdot|r\vec{B}|^2$$

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi)^2}\,\frac{a^2(t_e)\,\sin^2\theta}{c^3}$$

Picture - Polar Plot



- Only transverse currents matter, No radiation in same direction as particle
- Polarization. $\vec{B}$ is $\perp$ to $\vec{n}$ and $\vec{a}$

Larmor Pg. 3

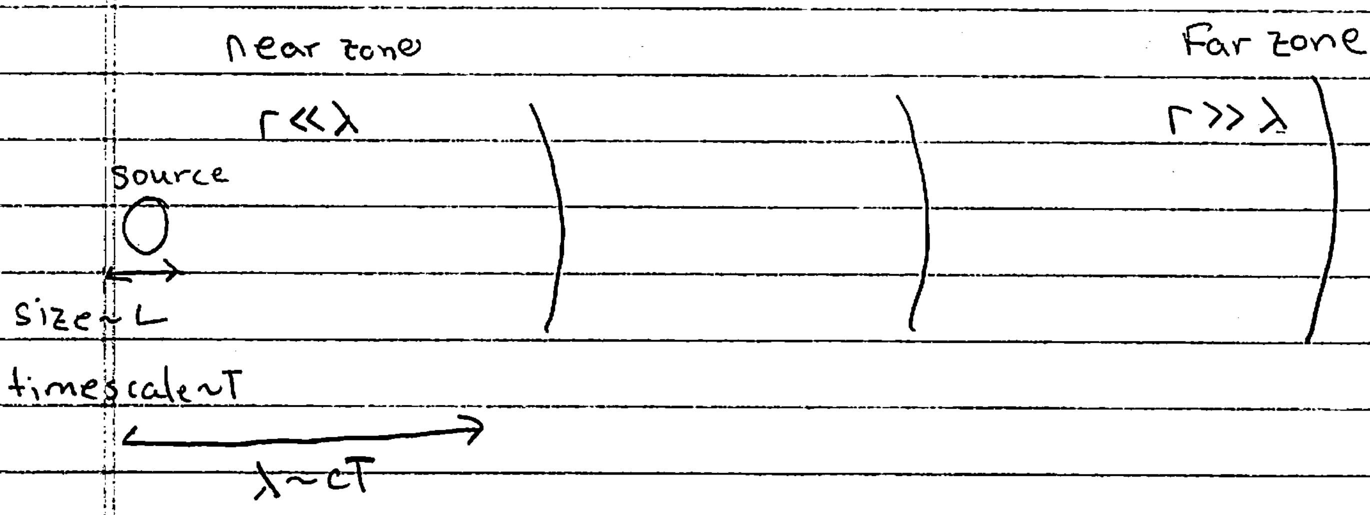
Then

$$P = \int d\Omega \, \frac{dP}{d\Omega} = \boxed{\frac{e^2}{4\pi} \cdot \frac{2\, a^2(t_e)}{3\, c^3} = P}$$

- Power passing through spherical shell at radius r at time t due to acceleration at time $t - \frac{r}{c} = t_e$
- A beautiful and simple result worth memorizing

Cartesian Multipole Expansion:

- Valid when $L \ll \lambda$:



The vector potential is:

$$\vec{A} = \frac{1}{4\pi r} \int_{r_0} \frac{\vec{J}}{c}\left(t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c}, r_0\right)$$

For small systems we have three equivalent statements:

$$\frac{r_0}{ct} \ll 1 \quad \text{or} \quad \underbrace{\frac{L}{\lambda} \ll 1}_{\text{small}} \quad \text{or} \quad \underbrace{\frac{L}{T} \ll c}_{\text{non-rel}}$$

Can Expand:

$$\vec{J}\left(t - \frac{r}{c} + \frac{\vec{n}\cdot\vec{r}_0}{c}\right) = \vec{J}\left(t - \frac{r}{c}\right) + \left(\frac{n\cdot r_0}{c}\right)\frac{\partial \vec{J}}{\partial t}\left(t - \frac{r}{c}\right) + \left(\frac{\vec{n}\cdot\vec{r}_0}{c}\right)^2 \frac{\partial^2 J}{\partial t^2}\left(t - \frac{r}{c}\right)$$

electric dipole

magnetic dipole approx ~~of electric~~ quadrupole $O\left(\frac{L}{\lambda}\right)$

higher ~~order~~ $O\left(\left(\frac{L}{\lambda}\right)^2\right)$

Electric Dipole Radiation : E1

$$\vec{A}_{rad} = \frac{1}{4\pi r} \int_{r_0} \frac{\vec{J}}{c}\left(t - \frac{r}{c}, r_0\right) \qquad t_e \equiv t - \frac{r}{c}$$

Using: $\partial r_0^{\ell} / \partial r_0^{i} = \delta^{\ell}_{i}$ we find

$$J^{\ell}(t_e, \vec{r}_0) = \underbrace{\frac{\partial}{\partial r_0^i}\left(J^i r_0^{\ell}\right)}_{\text{total divergence}} - \underbrace{\frac{\partial J^i}{\partial r_0^i}}_{-\frac{\partial \rho}{\partial t_e}(t_e) = -\frac{\partial \rho}{\partial t}(t_e)} r_0^{\ell}$$

So

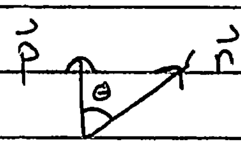
$$\int_{r_0} \vec{J}(t_e, r_0) = \int \frac{\partial}{\partial t} \rho(t_e, \vec{r}_0)\, \vec{r}_0 = \frac{\partial}{\partial t} \vec{p}\left(t - \frac{r}{c}, r_0\right)$$

Where the dipole moment, $\vec{p}\left(t - \frac{r}{c}\right) = \int_{r_0} \rho\left(t - \frac{r}{c}, r_0\right) \vec{r}_0$.

Then

$$\vec{A}^{E1}_{rad} = \frac{1}{4\pi r} \frac{1}{c} \frac{\partial}{\partial t} \vec{p}\left(t - \frac{r}{c}\right) \quad \text{and}$$

$$\vec{B} = -\vec{n} \times \frac{1}{c} \frac{\partial \vec{A}_{rad}}{\partial t} \propto \ddot{p}(t_e) \sin\theta$$

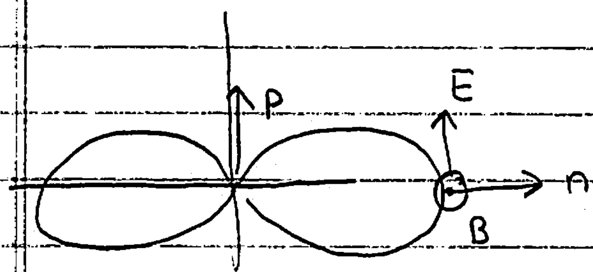


Electric Dipole pg. 2

Then the power radiated is

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2} \frac{\ddot{p}^2}{c^3} \sin^2\theta$$ and the total power is

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{1}{4\pi} \frac{2}{3} \frac{\ddot{p}^2}{c^3}$$



← Examine polarization

For a harmonic field $p(t-\frac{r}{c}) = \vec{p}_0 e^{-i\omega(t-r/c)}$, then that the time averaged power is

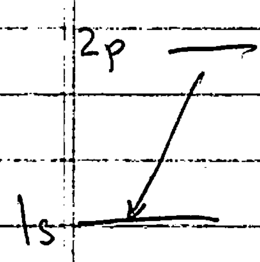
$$P = \left\langle \frac{dE}{dt} \right\rangle_{time} = \frac{1}{4\pi} \frac{2}{3c^3} \omega^4 \frac{p_0 \cdot p_0^*}{2}$$

← ~~factor of two~~ comes from time ave.

$$\boxed{P = \frac{1}{4\pi} \frac{\omega^4}{3c^3} |\vec{p}_0|^2}$$

For atomic lines, the decay rate Γ:

$$\Gamma = \frac{1}{\omega_0} \frac{dE}{dt} \propto \frac{1}{\omega_0} \omega_0^4 \propto \omega_0^3$$



Lifetime $\equiv 1/\Gamma$ of excited state is inversely propto ω_0^3!

Magnetic Dipole (M1) + Electric Quadrupole (E2) radiation

$$\vec{A}_{rad} = \frac{1}{4\pi r}\int_{r_0} \frac{\vec{n}\cdot\vec{r}_0}{c}\,\frac{\partial\vec{J}}{\partial t}\left(t-\frac{r}{c}, r_0\right)/c$$

$$A^j_{rad} = \frac{n_i}{4\pi r c}\int_{r_0} r_0^i\,\frac{\partial J^j}{\partial t}(r_0)/c$$

As always the tensor $r_0^i \frac{\partial J^j}{\partial t}$ should be broken up into its irreducible components and analyzed separately

$$r_0^i\,\partial_t J^j = \underbrace{\frac{1}{2}\left(r_0^i\partial_t J^j + r_0^j\partial_t J^i - \frac{2}{3}\delta^{ij}\right)}_{\text{Quadrupole rad}} + \underbrace{\frac{1}{2}\varepsilon^{ijk}(\vec{r}_0\times\partial_t\vec{J})_k}_{\text{mag dipole}} + \underbrace{\frac{1}{3} r_i\cdot\partial_t J^i}_{\text{gives no radiation}}$$

a monopole doesn't radiate

We will analyze each piece separately. First focus on the mag. dipole only.

$$A^j_{rad} = \frac{1}{4\pi r c}\int_{r_0} -\frac{1}{2}\varepsilon^{jik}\,n_i\,(\vec{r}_0\times\partial_t\frac{\vec{J}}{c})_k$$

$$\vec{A} = -\frac{1}{4\pi r}\,\frac{\vec{n}}{c}\times\frac{1}{2}\int_{r_0}\vec{r}_0\times\frac{\partial\vec{J}}{\partial t}/c$$

$$\vec{A}_{rad} = -\frac{1}{4\pi r}\,\frac{\vec{n}}{c}\times\dot{\vec{m}}\left(t-\frac{r}{c}\right)$$

we defined the magnetic dipole moment

$$\vec{m} = \frac{1}{2}\int_{r_0}\vec{r}_0\times\frac{\vec{J}(t_e,\vec{r}_0)}{c}$$

Magnetic Dipole pg. 2

Aside: Thus for a harmonic field, $\vec{m}(t-\frac{r}{c}) = e^{-i\omega t + ikr} m_0$, $k = \omega/c$

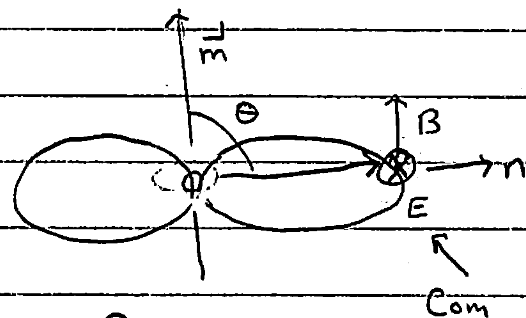
find a classic outgoing spherical wave:

$$\vec{A} = -\frac{1}{4\pi}\frac{\vec{n} \times m_0}{c}\frac{e^{-i\omega t + ikr}}{r}$$

Continue. The radiated power is, since $\vec{E} = \vec{n} \times \vec{n} \times \frac{1}{c}\partial_t \vec{A}$,

$$\frac{dP}{d\Omega} = \frac{1}{16\pi^2}\frac{1}{c^3}\left| \vec{n} \times \ddot{\vec{m}}(t-\frac{r}{c}) \right|^2$$

$$\frac{dP}{d\Omega} = \frac{\ddot{m}^2 \sin^2\theta}{16\pi^2 c^3}$$



So the angular distribution of power is the same as the electric case, but the polarization is reversed. Again we see for harmonic fields

$$m(t-\frac{r}{c}) = m_0 e^{-i\omega t + i\frac{\omega}{c}r}$$

The power is $P \propto \omega^4$.

Relative Strengths of E1 vs M1 radialtion

- The formulas are very similar. If the system has both an electric dipole and a magnetic diopole then both contribute to the radiated power
- Lets compare the E-dipole and M-dipole:

$$\vec{p} \sim \int_{r_0} \rho \vec{r} \sim e\vec{L}$$

$$\vec{m} \sim \frac{IA}{c} \sim \frac{(e)}{Tc} L^2 \sim eL\frac{V}{c}$$

typical vebcity of chrged particles

units

So

$$\frac{m}{p} \sim \frac{V}{c}$$

And thus the power radiated by the magnetic dipole moment is less than for the electric dipole

$$\frac{P^{M1}}{P^{E1}} \propto \frac{m^2}{p^2} \propto \left(\frac{V}{c}\right)^2$$

So the power radiated by the magnetic dipole is smaller