Relativity

- The speed of light is constant for all inertial observers. Nothing travels faster than c.
- The laws of physics are the same for all observers.

**Earth** \[ \frac{d\mathbf{p}}{dt} = q \mathbf{E} + q \mathbf{v} \times \mathbf{B} \] (the only thing relativity changes)
- \( \nabla \cdot \mathbf{E} = \rho \)
- \( \frac{1}{c^2} \frac{d\mathbf{E}}{dt} + \nabla \times \mathbf{B} = \mathbf{j} \)
- \( \nabla \cdot \mathbf{B} = 0 \)
- \( \frac{1}{c^2} \frac{d\mathbf{B}}{dt} + \nabla \times \mathbf{E} = 0 \)

Andromeda measures the same equations:

2g. \[ \frac{d\mathbf{p}}{dt} = q \mathbf{E} + q \mathbf{v} \times \mathbf{B} \] \[ \frac{d\mathbf{E}}{dt} + \mathbf{v} \cdot \nabla \mathbf{E} = 0 \]

Relativity relates \((t, \mathbf{x}, \mathbf{E}, \mathbf{p}, \mathbf{j})\) to \((\mathbf{t}, \mathbf{x}, \mathbf{E}, \mathbf{p}, \mathbf{j})\)

- Our goal is to introduce a notation to make this manifest.
Transformation of Coordinates

We describe physics as a sequence of events labelled by space-time coords \( x^m = (x^0, x_1, x_2, x_3) \)

\( \varepsilon (ct, x') \)

Each observer sets up his own coordinate system

An observer moving with speed \( v \) relative to the first makes measurement of the same events. The order of events is respected only if causally connected.

Light \( \leq \) speed of light const
We look for a linear change of coordinates which leaves the trajectory of light at fixed speed \( c = x/t \).

\[ -c^2 t^2 + x^2 = -c^2 t'^2 + x'^2 \]

So \( x \to x^\prime = L^\prime(x) = L^\prime_v(x) \).

\[
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
= 
\begin{pmatrix}
  L(v) & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
\]

Properties:

\[ L(-v) L(v) = 1 \]

\[ L(-v) = L^{-1}(v) \]

\[ L(v_2) L(v_2) = L(v_3) \]

This is a group of transformations, the Lorentz Group. With \( \star \) and \( \star \star \) find

\[
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
= 
\begin{pmatrix}
  \gamma - \gamma \beta & 0 & 0 & 0 \\
  -\gamma \beta & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
\]

\[ \gamma = \frac{1}{\sqrt{1-\beta^2}} \]

\[ \beta \equiv \frac{v}{c} \]
Parametrize the frame velocity $\frac{v}{c}$ with rapidity

$$\frac{v}{c} = \tanh \gamma$$

So

$$\begin{cases} 
\gamma = \cosh \gamma \\
\beta = \sinh \gamma
\end{cases} \quad \frac{x^0}{x} = x^0 \cosh \gamma - x^1 \sinh \gamma
\quad x^1 = -x^1 \sinh \gamma - x^0 \cosh \gamma$$

Four vectors

- We denote $x^\mu = (x^0, x^1, x^2, x^3) = (-ct, \mathbf{x})$

Then

$$x_\mu x^\mu = x_0 x^0 + x_i x^i = -(ct)^2 + \mathbf{x} \cdot \mathbf{x}$$

is invariant. Since under a change of frame

$$X^\mu = L^\mu_\nu x^\nu$$

We find

$$x_\mu = x^\nu (L^{-1})^\mu_\nu \quad \text{or} \quad \left( \begin{array}{cccc} x^0 & x^1 & x^2 & x^3 \end{array} \right) = \left( \begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \end{array} \right) \left( L^{-1} \right)$$
Then

\[
X_{\mu} X^{\mu} = (x_{0}, x_{\nu}) \begin{pmatrix} x^{0} \\ x^{i} \end{pmatrix} = (x_{0}, \ldots, x^{0}) \begin{pmatrix} L^{-1} \\ L \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{i} \end{pmatrix} = (x_{0}, \ldots, x^{0}) \begin{pmatrix} x^{0} \\ x^{i} \end{pmatrix} = x_{\mu} x^{\mu}
\]

In general define a four vector \( A^{\mu} = (a^{0}, \vec{a}) \) as a quantity that transforms like:

\[
A^{\mu} = L^{\mu}_{\nu} A^{\nu} \quad \text{(contravariant components)}
\]

The \( n \) co-variant components \( A_{\mu} = (a_{0}, \vec{a}) = (-a^{0}, \vec{a}) \)

\[
A_{\mu} = A_{\nu} (L^{-1})^{\nu}_{\mu} \quad \text{transformation rule for lower indices}
\]

Then

\[
A_{\mu} A^{\mu} = -(a^{0})^{2} + \vec{a}^{2} \quad \text{is invariant}
\]

Similarly

\[
A_{\mu} B^{\mu} = a_{0} b^{0} + a ; b
\]

\[
= -a^{0} b^{0} + \vec{a} \cdot \vec{b} \quad \text{is invariant}
\]
Example - Relativistic Doppler Shift

A \ e^{-i\omega t + i\vec{k} \cdot \vec{x}} \quad \text{a wave with} \quad \omega = c k

Since the speed of light is constant

\[ \phi = -\omega t + \vec{k} \cdot \vec{x} = -\frac{\omega}{c} t + \vec{k} \cdot \vec{x} \]

must be constant. This shows that \( \frac{\omega}{c}, \vec{k} \) form a four vector

\[ \vec{k} \cdot \vec{x} = \frac{\omega}{c} x - \vec{k} \cdot \vec{x} \]

So \( \vec{k} \cdot \vec{x} = \frac{\omega}{c} x - \vec{k} \cdot \vec{x} \) is constant.
Velocity transformation rules and four velocity

\[ \hat{u} =? \]

\[ \hat{u} = \frac{d\hat{x}^\mu}{d\tau} \]

Use \[ x^n = \hat{u}^\mu x^\mu \]

\[ u'' = \frac{u'' - v}{1 + \hat{u}\hat{u}/c^2} \]

\[ u_\perp = \frac{u_\perp}{\gamma (1 + \hat{u}\hat{u}/c^2)} \]

Complicated because the numerator and denominator transform. Consider a particle moving, \( dx^\mu = (dx^0, dx^2) = dx^0 (1, \hat{\beta}^2) \)

\[ ds^2 = -(dx^0)^2 + dx^2 = dx^\alpha dx^\nu \]

\[ ds^2 = -(dx^0)^2 (1 - \hat{\beta}^2) \]

\[ ds^2 \] is invariant. In the instantaneous rest frame of a particle, \( ds^2 = -c^2 d\tau^2 + dx^2 \)
So one finds that $d\tau$ is a Lorentz invariant:

$$d\tau = dt \sqrt{1 - \beta^2} = \frac{dt}{\gamma} = d\tau$$

So define the four velocity: (distance per proper time)

$$U^\mu = \frac{dX^\mu}{d\tau} = \left( \gamma \frac{dx^0}{dt}, \gamma \frac{dx^i}{dt} \right)$$

$$U^0 = (\gamma c, \gamma \vec{u})$$ and $$U^\mu = L^\mu_{\nu} U^\nu$$

As we will see the energy and momentum are proportional to the four velocity.

Energy, momentum and velocity

- For a particle at rest

  $$E = mc^2$$

- For a particle moving slowly

  $$U = \frac{c^2 \frac{p}{E} = \frac{\vec{p}}{m}}{E}$$