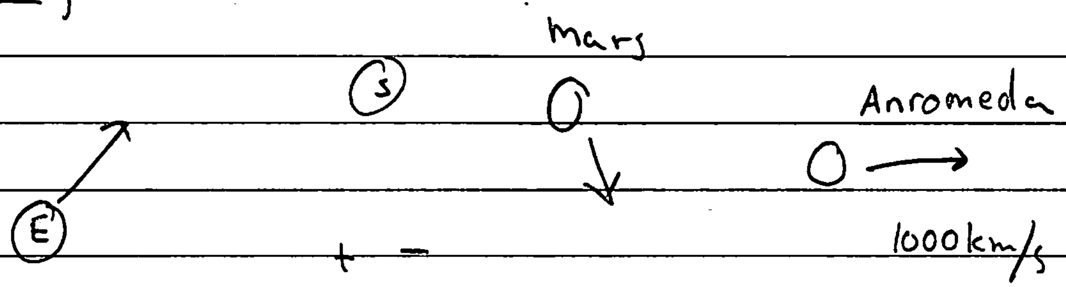
Relativity



- The speed of light is constant for all inertial observers. Nothing travels faster than c

- The Laws of physics are the same for all observers

Earth

$$\frac{dp}{dt} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

$$\nabla\cdot E = \rho$$

$$-\frac{1}{c}\frac{\partial E}{\partial t} + \nabla\times B = j$$

$$\nabla\cdot B = 0$$

$$\frac{1}{c}\frac{\partial B}{\partial t} + \nabla\times E = 0$$

the only thing relativity changes is the relation between $\vec{v}$ and p

$$\vec{v} = \frac{\vec{p}}{E/c^2}$$

Andromea measures the same equations:

eg. $$\frac{d\underline{p}}{d\underline{t}} = q\vec{\underline{E}} + q\frac{\underline{v}}{c}\times\vec{\underline{B}} \qquad \frac{\partial\underline{\rho}}{\partial\underline{t}} + \underline{\nabla}\cdot\underline{J} = 0$$

Relativity relates $(t, \vec{x}, \vec{E}, \vec{B}, \rho, \vec{J})$ to $(\underline{t}, \underline{x}, \vec{\underline{E}}, \vec{\underline{B}}, \underline{\rho}, \underline{J})$

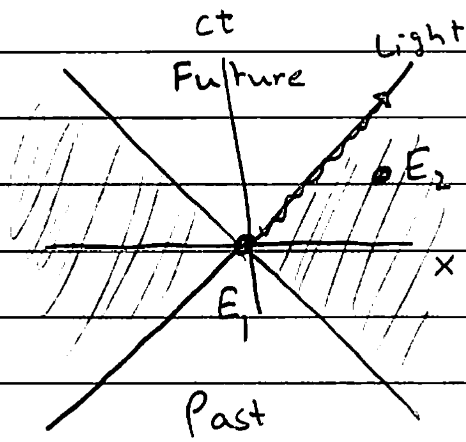
- Our goal is to introduce a notation to make this manifest.

Transformation of Coordinates

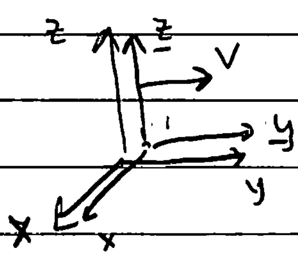
We describe physics as a sequence of events labelled by space-time coords $x^\mu = (x^0, x^1, x^2, x^3)$

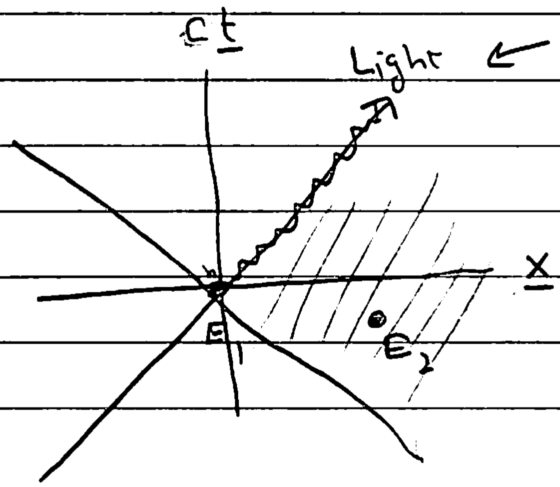
$$\equiv (ct, \vec{x})$$

Each observer set up his own coordinate system recording events



An observer moving with speed v relatives to the first makes measurement of the same events. The order of events is sacred only if causally connected





← speed of Light const

We look for a linear change of coordinates which leaves the trajectory of Light at fixed speed, $c = x/t$

☆ $$-c^2t^2 + \vec{x}^2 = -c^2\underline{t}^2 + \underline{\vec{x}}^2$$

So $$x \rightarrow \underline{x}^\mu = L^\mu{}_\nu(v)\, x^\nu .$$

$$\begin{pmatrix} \underline{x}^0 \\ \underline{x}^1 \\ \underline{x}^2 \\ \underline{x}^3 \end{pmatrix} = \begin{pmatrix} & & & \\ & L(v) & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} =$$

how space time coords of events in one frame K transform, to another frame $\underline{K}$ moving @ velocity v

Properties

$$L(-\vec{v})\,L(\vec{v}) = 1 \qquad\qquad L(-v) = L^{-1}(v)$$

☆☆

$$L(v_2)\;L(v_1) = L(v_3)$$

This is a group of transformations. The Lorentz Group. With ☆ and ☆☆ find

$$\begin{pmatrix} \underline{x}^0 \\ \underline{x}^1 \\ \underline{x}^2 \\ \underline{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \beta \equiv \frac{v}{c}$$

Parametrize the frame velocity $\frac{V}{c}$ with rapidity

$$\frac{V}{c} = \tanh \eta$$

So

$$\left.\begin{aligned} \gamma &= \cosh \eta \\ \gamma\beta &= \sinh \eta \end{aligned}\right\} \qquad \begin{aligned} \underline{x}^0 &= x^0 \cosh \eta - x^1 \sinh \eta \\ x^1 &= - x^1 \sinh \eta - x^0 \cosh \eta \end{aligned}$$

Four vectors

- We denote $x_\mu = (x_0, x_1, x_2, x_3) = (-ct, \vec{x})$

Then

$$X_\mu X^\mu = x_0 x^0 + x_i x^i = -(ct)^2 + \vec{x} \cdot \vec{x}$$

is invariant. Since under a change of frame

$$\underline{X}^\mu = L^\mu{}_\nu X^\nu$$

We find

$$\underline{X}_\mu = X_\mu (L^{-1})^\mu{}_\nu \quad \text{or}$$

$$(\underline{x}_0\ \underline{x}_1\ \underline{x}_2\ \underline{x}_3) = (x_0\ x_1\ x_2\ x_3) \begin{pmatrix} L^{-1} \end{pmatrix}$$

Then

$$\underline{x}_\mu \underline{x}^\mu = \underbrace{(\underline{x}_0 \cdots \underline{x}_3)}_{\text{Lower components}} \overset{\text{upper components}}{\begin{pmatrix} \underline{x}^0 \\ \vdots \\ \underline{x}^3 \end{pmatrix}} = (x_0 \cdots)\begin{pmatrix} L^{-1} \end{pmatrix}\begin{pmatrix} L \end{pmatrix}\begin{pmatrix} x^0 \\ \vdots \end{pmatrix}$$

$$= (x_0 \cdots)\begin{pmatrix} x^0 \\ \vdots \end{pmatrix}$$

$$= x_\mu x^\mu$$

In general define a four vector $A^\mu = (a^0, \vec{a})$ as a quantity that transforms like

$$\underline{A}^\mu = L^\mu{}_\nu A^\nu \qquad \text{(contravariant components)}$$

The co-variant components $A_\mu = (a_0, \vec{a}) = (-a^0, \vec{a})$

$$\underline{A}_\mu = A_\mu (L^{-1})^\mu{}_\nu \quad \leftarrow \text{transformation rule for lower indices}$$

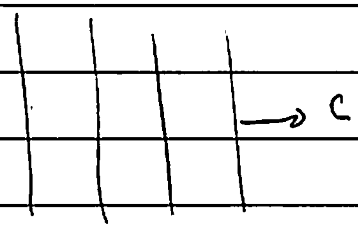
Then

$$A_\mu A^\mu = -(a^0)^2 + \vec{a}^2 \quad \text{is invariant}$$

Similarly $A_\mu B^\mu = a_0 b^0 + a_i b^i$

$$= -a^0 b^0 + \vec{a}\cdot\vec{b} \quad \text{is invariant}$$

Example - Relativistic Dopler Shift



$A e^{-i\omega t + i\vec{k}\vec{x}}$ a wave with $\omega = ck$

Since the speed of Light is constant

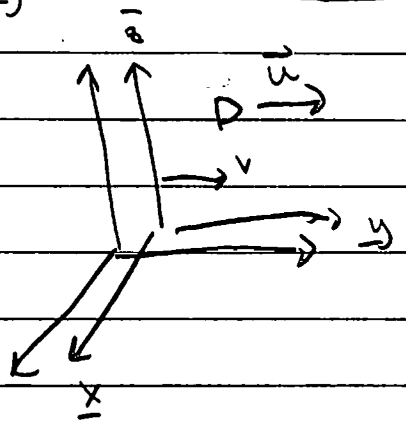
$$\phi = -\omega t + \vec{k}\cdot\vec{x} = -\underline{\omega}\,\underline{t} + \underline{\vec{k}}\cdot\underline{\vec{x}}$$

must be constant. This shows that $(\frac{\underline{\omega}}{c}, \vec{k})$ form a four vector

$$K^{\mu} = \left(\frac{\omega}{c}, \vec{k}\right)$$

So $K_{\mu} X^{\mu} = \underline{K}_{\mu}\,\underline{X}^{\mu} = -\omega t + \vec{k}\cdot\vec{x}$ is const.

Velocity transformation rules of Four velocity



$\underline{\vec{u}} = ?$

$$\vec{u} = \frac{d\vec{x}}{dt} \qquad \underline{\vec{u}} = \frac{d\underline{\vec{x}}}{dt}$$

numerator

denominator

Use. $\underline{x}^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}$

$$\underline{u}'' = \frac{u'' - v}{1 + \vec{u}\cdot\vec{v}/c^2} \qquad \text{etc.}$$

$$\underline{u}_{\perp} = \frac{u_{\perp}}{\gamma\,(1 + \vec{u}\cdot\vec{v}/c^2)}$$

Complicated, because both the numerator and denominator transform. Consider a particle moving, $dx^{\mu} = (dx^0, d\vec{x}) = dx^0\,(1, \vec{\beta}_u)$

$$ds^2 = -(dx^0)^2 + d\vec{x}^2 = dx_{\mu}\, dx^{\mu}$$

$$ds^2 = -(dx^0)^2\,(1 - \vec{\beta}_u^{\,2})$$

ds^2 is invariant. In the instantaneous rest frame of particle $ds^2 = -c^2 d\tau^2 + \cancel{d\vec{x}^2}$ 0

propper time of particle

So one finds that $d\tau$ is a lorentz invariant

$$d\tau = dt\sqrt{1-\beta_u^2} = \boxed{\frac{dt}{\gamma_u} = d\tau}$$

So define the four velocity: (distance per propper time)

$$\boxed{U^\mu = \frac{dX^\mu}{d\tau}} = \left(\gamma_u \frac{dx^0}{dt}, \gamma_u \frac{d\vec{x}}{dt}\right)$$

$$\boxed{U^\mu = (\gamma_u c, \gamma_u \vec{u})} \quad \text{and} \quad \boxed{U^\mu = L^\mu{}_\nu U^\nu}$$

As we will see the energy and momentum are proportional to the four velocity.

Energy momentum and velocity

- For a particle at rest

$$E = mc^2$$

- For a particle moving slowly

$$U = \frac{c^2\vec{p}}{E} \simeq \frac{\vec{p}}{m}$$