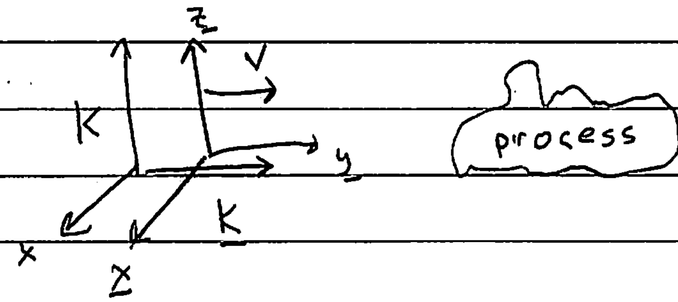
Last Time

(1) Introduced Relativity



- All laws of physics are the same

$$\underbrace{\frac{d\underline{p}}{d\underline{t}} = q\,\underline{\vec{E}} + \frac{\underline{\vec{v}}}{c}\times\underline{\vec{B}}}_{\text{Laws of phsics in K-frame}} \qquad \Bigg| \qquad \underbrace{\frac{dp}{dt} = q\,\vec{E} + \frac{\vec{V}}{c}\times\vec{B}}_{\text{Laws of Phys in } \underline{K} \text{ frame}}$$

- Speed of Light is const

(2) To relate $\underline{t}, \underline{x}$ to t, x we constructed a change of coords which leaves light speed the same

$$\underline{x}^{\mu} = L^{\mu}{}_{\nu}(v)\, x^{\nu} \qquad L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\underline{x}_{\mu} = x_{\nu}\,(L^{-1})^{\mu}{}_{\nu}$$

So

$$x_{\mu} x^{\mu} = -(ct)^2 + \vec{x}^{\,2} \quad \text{is invariant}$$

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(4) Also introduced the concept of a four vector. $A^\mu = (a^0, \vec{a})$. A four vector is a set of physical quantities which transform under change of frame (observer)

$$A^\mu = L^\mu{}_\nu(v)\, A^\nu$$

The dot product

$$A \cdot B = A_\mu B^\mu = a_0 b^0 + a_i b^i$$

$$= -a^0 b^0 + \vec{a} \cdot \vec{b}$$

(5) Formalize the transition between upper and lower indices

$$A_\mu = g_{\mu\nu} A^\nu \qquad g_{\mu\nu} \approx \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

metric tensor

$g_{00} = -1$ $g_{11} = g_{22} = g_{33} = 1$ all others zero

Then

$$A \cdot B = A^\mu g_{\mu\nu} B^\nu$$ is lorentz invariant

Formalize Raising and Lowering pg. 2

Indices are raised with

$A^\mu = g^{\mu\nu} A_\nu$ so $g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$A^\mu = \underbrace{g^{\mu\rho} g_{\rho\sigma}}_{} A^\sigma$

$\delta^\mu{}_\sigma = g^\mu{}_\sigma$ = identity matrix $= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Then $\underline{g}_{\mu\nu}$ same as $g_{\mu\nu}$ → thats relativity

$\underline{A}^\mu \underline{g}_{\mu\nu} \underline{B}^\nu = A^\mu g_{\mu\nu} B^\nu$

Matrices $\underline{A}^T g \underline{B} = A^T g B$ $\qquad \underline{A} = L A$

$A^T L^T g L B = A^T g B$ $\qquad \underline{A}^T = A^T L^T$

So require

$L^T g L = g$ or $g = (L^{-1T}) g L^{-1} = \underline{g}$

Or since $(L^T)^{-1} = (L^{-1})^T$ and $g^{-1} = g$

$g L g = (L^{-1})^T$ or $\boxed{g L^T g = L^{-1}}$

$$\boxed{\underbrace{g_{\rho\mu} L^\mu{}_\nu g^{\nu\sigma}}_{\equiv L_\rho{}^\sigma} = (L^{-1T})_\rho{}^\sigma}$$

Formalize Raising and Lowering Pg. 3

with this cryptic but standard notation

$$\underline{A}_{\nu} = A_{\mu} (L^{-1})^{\mu}{}_{\nu}$$

$$= (L^{-1T})_{\nu}{}^{\mu} A_{\mu}$$

$$\underline{A}_{\nu} = L_{\nu}{}^{\mu} A_{\mu}$$

Excereise:

- A Tensor transforms as

$$\bigstar \quad \underline{T}^{\mu\nu} = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} T^{\rho\sigma}$$

- Writing $T^{\mu}{}_{\nu}$ as a matrix $(T^{\mu}{}_{\nu})$ Show starting from $\bigstar$ that

$$\boxed{(\underline{T}^{\mu}{}_{\nu}) = (L)^{\mu}{}_{\rho} (T^{\rho}{}_{\sigma}) (L^{-1})^{\sigma}{}_{\nu}}$$

Or

$$\underline{T}^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} L_{\nu}{}^{\sigma} T^{\rho}{}_{\sigma}$$

Solution: Lowering ν by multiplying ω $g_{\nu'\nu}$:

$$\underline{T}^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} L_{\nu\sigma} T^{\rho\sigma} = L^{\mu}{}_{\rho} L_{\nu}{}^{\sigma} T^{\rho}{}_{\sigma}$$

$$= L^{\mu}{}_{\rho} (L^{-1T})_{\nu}{}^{\sigma} T^{\rho}{}_{\sigma}$$

$$\underline{T}^{\mu}{}_{\nu} = L^{\mu}{}_{\rho} T^{\rho}{}_{\sigma} (L^{-1})^{\sigma}{}_{\nu}$$

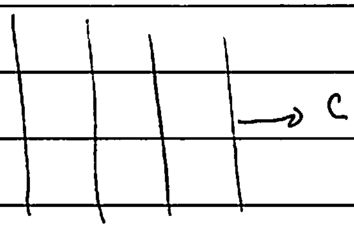
Reverse order of (σ, ν) and get rid of transpose

Can note: $A_{\mu} B^{\mu} = A_{\mu} \delta^{\mu}{}_{\sigma} B^{\sigma}$

$$= A_{\mu} g^{\mu\nu} g_{\nu\sigma} B^{\sigma} = A^{\nu} B_{\nu}$$

$$= A^{\mu} B_{\mu}$$

Example - Relativistic Dopler Shift



$A e^{-i\omega t + i\vec{k}\vec{x}}$: a wave with $\omega = ck$

Since the speed of Light is constant

$$\phi = -\omega t + \vec{k}\cdot\vec{x} = -\underline{\omega}\,\underline{t} + \underline{\vec{k}}\cdot\underline{\vec{x}}$$

must be constant. This shows that $(\frac{\omega}{c}, \vec{k})$ form a four vector

$$K^{\mu} = \left(\frac{\omega}{c}, \vec{k}\right)$$

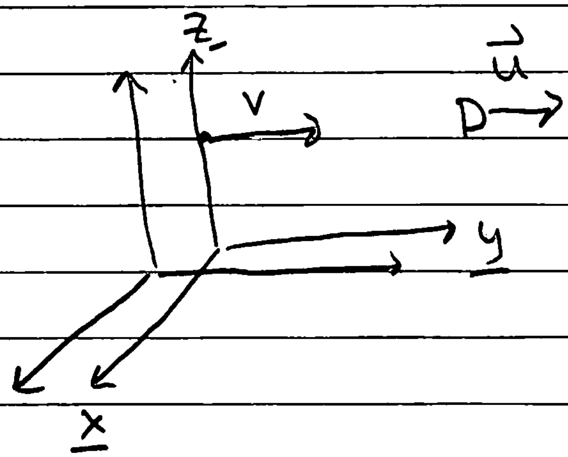
So $\quad K_{\mu} X^{\mu} = \underline{K}_{\mu}\,\underline{X}^{\mu} = -\omega t + \vec{k}\cdot\vec{x}$ is const.

- So if you know the frequency in one frame can find the frequency in another

- Then K^2 is lightlike (means $K^2 = 0$)

$$K^2 = -\left(\frac{\omega}{c}\right)^2 + \vec{k}^{\,2} = 0 \quad \text{in all frames}$$

Velocity Transformation Rule



$\underline{\vec{u}} = ?$

$\vec{u} = \dfrac{d\vec{x}}{dt}$ and $\underline{\vec{u}} = \dfrac{d\,\underline{\vec{x}}}{d\underline{t}}$

Use $\underline{x}^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}$ Find

$$\underline{u}^{\parallel} = \frac{u^{\parallel} - v}{(1 - \vec{u}\cdot\vec{v}/c^2)} \qquad \underline{u}_{\perp} = \frac{u_{\perp}}{\gamma(1 - \vec{u}\cdot\vec{v}/c^2)}$$

This is complicated because both the numerator and denominator transform

$$\frac{dx}{dt} \begin{array}{l} \leftarrow \text{numerator} \\ \leftarrow \text{denominator} \end{array}$$

Propper Time

- Consider a particle moving with velocity $\vec{u}$

$$dx^\mu = (dx^0, d\vec{x}) = dx^0 (1, \beta_u)$$

$$\beta_u = \frac{\vec{u}}{c} = \frac{d\vec{x}}{dx^0} \leftarrow \text{put } \beta_u = \frac{u}{c} \text{ and } \gamma_u = \frac{1}{\sqrt{1-(u/c)^2}}$$

to distinguish from $\beta = v/c$ and $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$

- Then:

$$ds^2 \equiv dx_\mu dx^\mu = -c^2 dt^2 (1 - \beta_u)^2$$

is invariant. In the rest frame of particle have

$$ds^2 = -c^2 d\tau^2 \leftarrow \text{propper time}$$

So

$$d\tau = dt\sqrt{1-\beta_u^2} = \boxed{\frac{dt}{\gamma_u} = d\tau}$$

- Define the four velocity, as the distance per propper time

$$U^\mu = \frac{dx^\mu}{d\tau} \leftarrow \text{this is a four vector}$$

$$U^\mu = \gamma_u \frac{dx^\mu}{dt} = (\gamma_u c, \gamma_u \vec{u}) \qquad \frac{d\vec{x}}{dt} = \vec{u}$$

A preview of 4-momentum

We will see:

four velocity

$$\boxed{p^\mu = m u^\mu}$$

four momentum $\boxed{\left(\frac{E}{c}, \vec{p}\right) \equiv p^\mu}$

$$p^\mu p_\mu = m^2 u^\mu u_\mu \qquad u^\mu u_\mu = -c^2$$

$$= -m^2 c^2 \qquad -\gamma_u^2 c^2 + \gamma_u^2 u^2$$

Then:

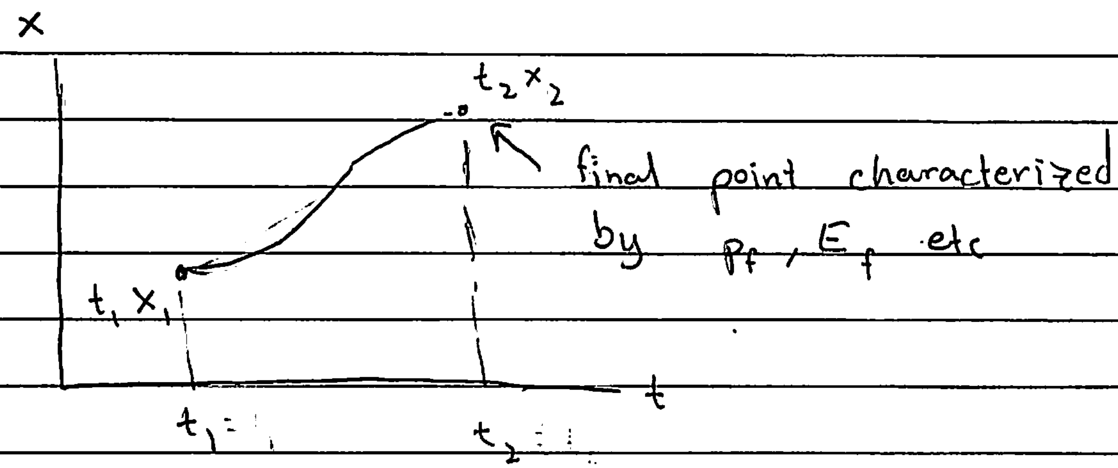
$$p^\mu p_\mu = -\left(\frac{E}{c}\right)^2 + p^2 = -m^2 c^2$$

$$\Longrightarrow \quad E = \sqrt{(mc^2)^2 + (cp)^2}$$

and

$$\boxed{\vec{u} = \frac{\vec{p}}{(E/c^2)}} = \frac{\gamma m \vec{u}}{(\gamma m c^2)/c^2} = \vec{u}$$

Energy and Momentum



Classical Dynamics mimizes the action

$$I = \int dt\, L[x, \dot{x}]$$

Look at the "on-shell action" or Hamilton-Jacobi function. Means you find the solution $X(t) = x_s(t)$ and evaluate I.

$$S(t_1 x_1; t_2 x_2) = I \Big|_{X(t) = x_s(t)} = \int dt\, L(x_s, \dot{x}_s(t))$$

How does S change as we change $t_2\, x_2$

Answer:

$$\delta S = -E_f\, \delta t_2 + p_f\, \delta x_2$$

Energy + Momentum pg. 2

Prf

$$S = \int L \, dt = \int (p \, d\dot{q} - H)\, dt$$

$$S = \int p \, dq - H \, dt$$

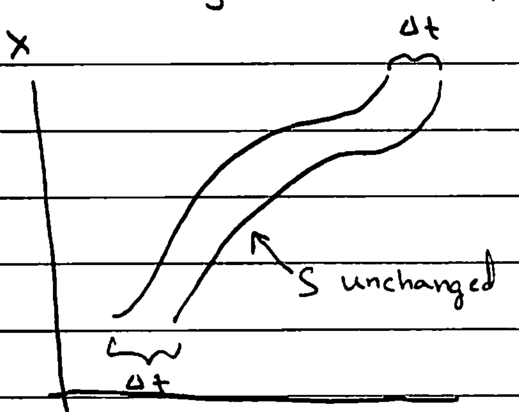
So if you change δq and δt

$$\delta S_2 = -E_f \, \delta t_2 + p_f \, \delta x_2$$

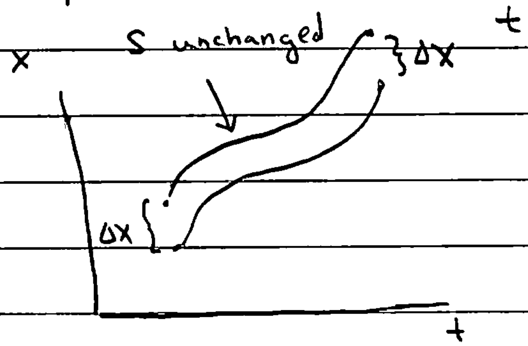
Similarly if you change t_1 and x_1

$$-\delta S_1 = -E_i \, \delta t_1 + p_i \, \delta x_1$$

Now for a system which has no preferred time origin (i.e. the resulting motion is independent of when you start), then the action integral is unchanged by changing the starting and final times by Δt, $\Delta S = 0$



Similarly If the system has no spatial origin, then the action is unchanged by shifting the initial and final points by Δx
$\Delta S = 0$



So

$$\Delta S = 0 = -(E_f - E_i)\Delta t + (p_f - p_i)\Delta x$$

And then energy and momentum are conserved

$$E_f = E_i$$

$$p_f = p_i$$

So energy and momentum is conservation is a consequence of the homogeneity of space-time. It should be true for all observers:

$$\delta S = -E\,\delta t + p\,\delta x = -\underline{E}\,\underline{\delta t} + \underline{p}\,\underline{\delta x}$$

Thus if $(c\delta t, \delta x)$ is a four vector

then, $P^\mu = \left(\frac{E}{c}, \vec{p}\right)$, should also be

a four vector, so we have the invariant,

$$P_\mu \delta x^\mu = -E\,\delta t + p\,\delta x$$

This guarrantees the energy and momentum conservation in all frames.

Energy and Momentum pg. 4

Using for small velocity $p = mv$

$$\begin{pmatrix} E/c \\ p \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E/c_{rest} \\ 0 \end{pmatrix}$$

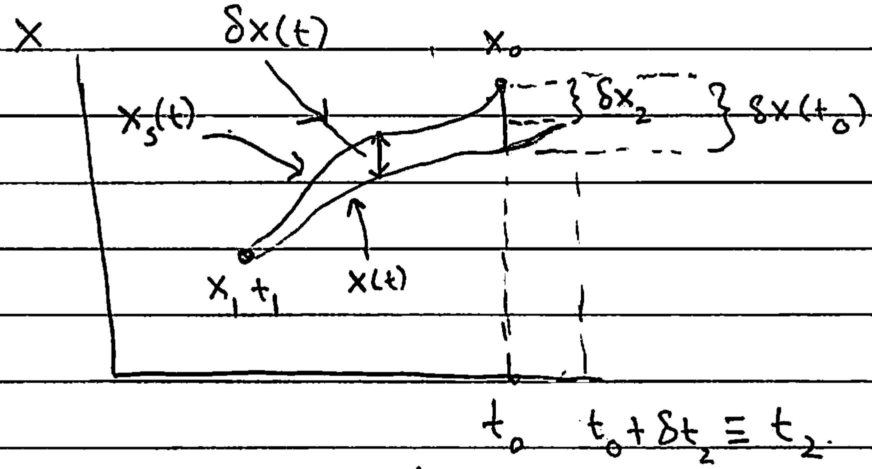
$$\begin{pmatrix} E/c \\ p \end{pmatrix} \simeq \begin{pmatrix} 1 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} E_{rest}/c \\ 0 \end{pmatrix}$$

$$p = \frac{E_{rest}}{c} \beta \simeq \frac{E_{rest}}{c^2} v$$

So must identify $\boxed{E_{rest} = mc^2}$, and

$$\boxed{p^\mu p_\mu = -m^2 c^2}$$

Excercise



$\delta x_2 \equiv \delta x(t_2)$

- Work entirely within the Lagrangian framework:

$$S = \int_{t_1}^{t_0+\delta t} L\left[x_s(t) + \delta x(t) \right]$$

Show that

$$\delta S = -E\,\delta t_2 + p\,\delta x_2 = \left(\frac{\partial S}{\partial t_2}\right)_{x_2} \delta t_2 + \left(\frac{\partial S}{\partial x}\right)_{t_2} \delta x_2$$

Note: $x(t) = x_s(t) + \delta x(t)$

$$x(t_0) = x_s(t_0) + \delta x(t_0)$$

$$\simeq x(t_0+\delta t) - \dot{x}(t_0+\delta t)\,\delta t = x_s(t_0) + \delta x(t_0)$$

$$\underbrace{\left[x(t_0+\delta t) - x_s(t_0) \right]}_{\delta x_2} - \dot{x}(t_2)\,\delta t = \delta x(t_0)$$

$$\delta x_2 - \dot{x}(t_2)\,\delta t = \delta x(t_0)$$

Solution to Excercise

$$\delta S = \mathcal{L}\cdot\delta t + \int_{t_1}^{t_0} \mathcal{L}(x+\delta x)$$

$$= \mathcal{L}\cdot\delta t + \int_{t_1}^{t_0}\left[\underbrace{\frac{\partial\mathcal{L}}{\partial\dot{x}}\,\partial_t\delta x}_{\text{integrate by parts}} + \frac{\partial\mathcal{L}}{\partial x}\delta x\right]$$

$$= \mathcal{L}\,\delta t + \frac{\partial\mathcal{L}}{\partial\dot{x}}\delta x\Big|_{t_1}^{t_0} + \int_{t_1}^{t_0} dt\,\underbrace{\left[-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} + \frac{\partial\mathcal{L}}{\partial x}\right]}_{=0 \text{ by EOM}}\delta x$$

$$= \mathcal{L}\,\delta t + \frac{\partial\mathcal{L}}{\partial\dot{x}}\delta x(t_0)$$

Now

$$\delta x_2 = \delta x(t_0) + \dot{x}(t_0)\,\delta t_2 \qquad \text{(see picture)}$$

So

$$\delta S = \left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{x}}\dot{x}\right)\delta t_2 + \frac{\partial\mathcal{L}}{\partial\dot{x}}\delta x_2$$

$$\delta S = -E\,\delta t + p\,\delta x$$

manifest

Covariant Formulation

① $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$ ← is a covariant vector

Also $\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$

$$\boxed{\partial_\mu \partial^\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \equiv \Box}$$ is invariant

② Then the continuity eqn:

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{J} = 0 \implies \frac{1}{c}\frac{\partial(c\rho)}{\partial t} + \nabla\cdot J = 0$$

So take $J^\mu = (c\rho, \vec{J})$ to be a four vector

$$\boxed{\partial_\mu J^\mu = 0}$$

③ Then the equations for the gauge potential read

$$-\Box\varphi = J^0/c$$

$$-\Box\vec{A} = \vec{J}/c$$

Together $\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\vec{A} = 0$

Covariance pg. 2

So in order to have a lorentz invariant theory take $(\varphi, \vec{A})$ to be a four vector

$$A^\mu = (\varphi, \vec{A})$$

Then the wave eq becomes

$$\boxed{-\Box A^\mu = J^\mu / c} \qquad \boxed{\underbrace{\partial_\mu A^\mu = 0}_{\text{Lorentz gauge condition}}}$$

(4) Now the fields

$$E^i = -\frac{\partial A^i}{c\,\partial t} - \frac{\partial \varphi}{\partial x_i} = \underbrace{\partial^0 A^i - \partial^i A^0}_{\equiv F^{0i}}$$

$$\equiv F^{0i}$$

$$B_k = (\nabla \times A)_k$$

$$\varepsilon^{ijk} B_k = \underbrace{\varepsilon^{ijk}\ \varepsilon_{klm}}_{(\delta^{il}\delta^{jm} - \delta^{jl}\delta^{km})} \partial_l A_m$$

$$\varepsilon^{ijk} B_k = \underbrace{\partial^i A^j - \partial^j A^i}_{\equiv F^{ij}}$$

Covariance pg. 3

So we can see

$$\boxed{F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}}$$

$$F^{\alpha\beta} = \begin{array}{cc} & \beta \longrightarrow \\ \alpha \downarrow & \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \end{array}$$

Note $F^{\alpha\beta}$ is a second rank anti-symmetric tensor and transforms in the following way

$$F^{\mu\nu} = L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} F^{\alpha\beta}$$

(5) Now the EOM (Part I)

$$\nabla \cdot E = \rho \implies -\partial_{l} F^{l0} = \rho$$

$$-\frac{1}{c}\partial_t E + \nabla \times B = J/c \implies -\left(\frac{\partial F^{0i}}{\partial x^0} + \frac{\partial F^{li}}{\partial x^l}\right) = J^i/c$$

Or $$\boxed{-\partial_{\alpha} F^{\alpha\beta} = \frac{J^{\beta}}{c}}$$

(6) The remaining two maxwell eqs follow from the fact that

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

Which we derived from:

$$\begin{aligned} \nabla\cdot B &= 0 \\ \nabla\times E &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \end{aligned} \quad \overset{\text{or}}{\Longrightarrow} \quad \begin{aligned} \nabla\cdot B &= 0 \\ -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} - \nabla\times E &= 0 \end{aligned}$$

Comparison with the other two maxwell eqs in the absence of currents

$$\left.\begin{aligned} \nabla\cdot E &= 0 \\ -\frac{\partial E}{c\,\partial t} + \nabla\times B &= 0 \end{aligned}\right\}$$

So if $\vec{E},\vec{B}$ are solutions then $\vec{E}' \to \vec{E}' = \vec{B}$ and $\vec{B} \to \vec{B}' = -\vec{E}$

Shows that we want to replace $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$

in the Field strength

$$\boxed{\tilde{\mathcal{F}}^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & B^x & B^y & B^z \\ \hline -\vec{B} & 0 & -E^x & E^y \\ & E^x & 0 & -E^z \\ & E^y & E^z & 0 \end{array}\right)}$$

So

$$\boxed{-\partial_\mu \tilde{\mathcal{F}}^{\mu\nu} = 0}$$

Then $\varepsilon^{\mu\nu\alpha\beta}$ implements $\vec{B} \to -\vec{E}$ and $\vec{E} \to \vec{B}$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

where
$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{for even perm of } 0,1,2,3 \\ -1 & \text{for odd perm of } 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

$$-\partial_\mu \tilde{F}^{\mu\nu} = 0$$

This can also be written in terms of F

$$\boxed{\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0}$$

This is the Bianchi - Identity

(7) Finally the Force Law

$$\frac{dE}{dt} = q\,\vec{E}\cdot\vec{v} = q\,F^{0i}\,v^{i}$$

$$\frac{dp^{i}}{dt} = q\,\vec{E} + \frac{\vec{v}}{c}\times\vec{B} = q\left(F^{i}{}_{0} + F^{i}{}_{j}\,\frac{v^{j}}{c}\right)$$

So

$$\boxed{\frac{dp^{\mu}}{d\tau} = \frac{q}{c}\,F^{\mu}{}_{\nu}\,u^{\nu}}$$