Last Time

Formulate E&M covariantly

(1) $\frac{dp^\mu}{d\tau} = q F^\mu{}_\nu \frac{u^\nu}{c}$ Force Law

- $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$ = four momentum $p^2 = p \cdot p$

$$= -\frac{(mc^2)^2}{c^2} = -\frac{E_{rest}^2}{c^2}$$

- $u^\mu = \frac{dx^\mu}{d\tau} = (\gamma_u c, \gamma_u \vec{u})$ = four velocity

= distance per propper time $u_\mu u^\mu = -c^2$

- $d\tau$ = propper time (time in RF of particle)

$= \frac{dt}{\gamma_u}$

- $F^{\mu\nu}$ = Field Strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\left(\begin{array}{c|ccc} 0 & E^x & E^y & E^z \\ \hline -E^x & 0 & B^x & -B^y \\ | & -B^x & 0 & B^z \\ | & B^y & -B^z & 0 \end{array}\right)$$

$E^i = F^{0i} = -F^{i0} = F^i{}_0$

$\varepsilon^{ijk} B_k = F^{ij} = -F^{ji}$

② Field Equations

• $-\partial_\mu F^{\mu\nu} = \frac{J^\nu}{c}$ $\qquad J^\nu = (c\rho, \vec{J})$

and

$-\Box A^\mu = J^\mu/c \qquad A^\mu = (\varphi, \vec{A})$

③ There are two others

$$\left.\begin{aligned} \nabla\cdot E &= 0 \\ -\frac{\partial E}{\partial t} + \nabla\times B &= 0 \end{aligned}\right\}$$ Current free eqs first two Eqs

$$\left.\begin{aligned} \nabla\cdot B &= 0 \\ -\frac{\partial B}{\partial t} - \nabla\times E &= 0 \end{aligned}\right\}$$ second two eqs: Same with $E \to B$, $\vec{B} \to -\vec{E}$

Define a dual field wich implements this replacement:

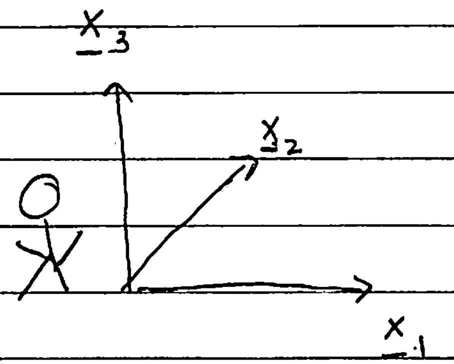
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \qquad \tilde{F}^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & \vec{B} & \\ \hline & 0 & -E^x & E^y \\ -\vec{B} & E^x & 0 & -E^z \\ & E^y & E^z & 0 \end{array}\right)$$

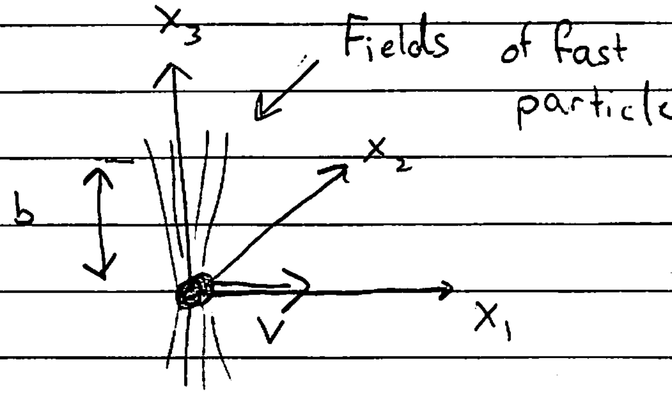
So

$$-\partial_\mu \tilde{F}^{\mu\nu} = 0$$

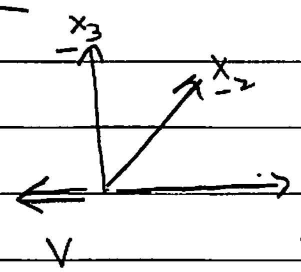
Fields of a fast moving Charged Particle

Person view

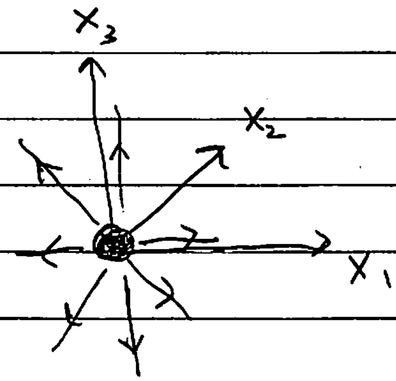




Particle View:



$\underline{F}^{\mu\nu}(x) = ?$



Know $F^{\alpha\beta}(x)$ = coulomb field

Given $F^{\mu\nu}(x)$ = coulomb field want to know $\underline{F}^{\mu\nu}(\underline{x})$ = the fields in frame of person. Then

$$\underline{F}^{\mu\nu}(\underline{x}) = L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} F^{\alpha\beta}(x)$$

Where for a boost in negative x-direction

$$L^{\mu}{}_{\nu}(v) = \begin{pmatrix} \gamma & +\gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

So given the field strength tensor $F^{\mu\nu}$ how does it transform.

$$\underline{F}^{0i} = L^0{}_\alpha L^i{}_\beta F^{\alpha\beta} \leftarrow \text{electric fields}$$

First look in the parallel direction x_1

a) $$\underline{E}^{\parallel} = L^0{}_\alpha L^1{}_\beta F^{\alpha\beta}$$

$$= L^0{}_0 L^1{}_1 F^{01} + L^0{}_1 L^1{}_0 F^{10}$$

$$\underline{E}^{\parallel} = \underbrace{(\gamma^2 - \gamma^2\beta^2)}_{=1} F^{01}$$

$$E^{\parallel} = E^{\parallel}$$

Similarly the perp direction

$$F^{02} = L^0{}_\alpha L^2{}_\beta F^{\alpha\beta}$$

$$= L^0{}_0 L^2{}_2 F^{02} + L^0{}_1 L^2{}_2 F^{12}$$

$$\underline{E}^2 = \gamma E^2 + \gamma\beta B^3$$

So

$$\boxed{\begin{array}{ll} \underline{\vec{E}}^{\parallel} = \vec{E}^{\parallel} & \underline{\vec{B}}^{\parallel} = B^{\parallel} \\ \underline{\vec{E}}_{\perp} = \gamma \vec{E}_{\perp} + \gamma \vec{\beta} \times \vec{B}_{\perp} & \underline{\vec{B}}_{\perp} = \gamma \vec{B}_{\perp} - \gamma \beta \times \vec{E}_{\perp} \end{array}}$$

① Looks a lot like coordinates transform
but is the transverse pieces which get boosted

② The trasformation for $\vec{B}$ is the dual of $\vec{E}$
$E \rightarrow B$, $\vec{B} \rightarrow -\vec{E}$

③ In the original frame $\vec{B} = 0$ so
we find

$$\boxed{\underline{\vec{B}} = -\vec{\beta} \times \underline{\vec{E}}}$$

most often used for non rel probs where
$\vec{B}^{(0)} \simeq 0$ and then $\vec{B}^{(1)} = -\beta \times \vec{E}^{(0)}$

Now the charge

$$E_1(x) = \frac{q}{(x^i x_i)^{3/2}}\, x_1 = E^{\parallel}(x)$$

$$E_2(x) = \frac{q}{(x^i x_i)^{3/2}}\, x_2 = (\vec{E}_\perp(x))_2$$

So we have under boost

$$\left.\begin{aligned} \underline{x}^\mu &= L^\mu{}_\nu\, x^\nu \\ (L^{-1})^\nu{}_\mu\, \underline{x}^\mu &= x^\nu \end{aligned}\right\} \quad \begin{aligned} x_1 &= \gamma(\underline{x} - v\underline{t}) \\ x_2 &= \underline{x}_2 \\ x_3 &= \underline{x}_3 \end{aligned}$$

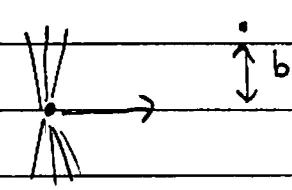
$$b^2 = x_2^2 + x_3^2$$

$$\vec{b} = (x_2, x_3)$$

impact parameter

So Fields

$$\underline{E}^\parallel(\underline{x}) = \frac{e\gamma(\underline{x} - v\underline{t})}{(b^2 + \gamma^2(\underline{x} - v\underline{t})^2)^{3/2}} = E^\parallel(x)$$

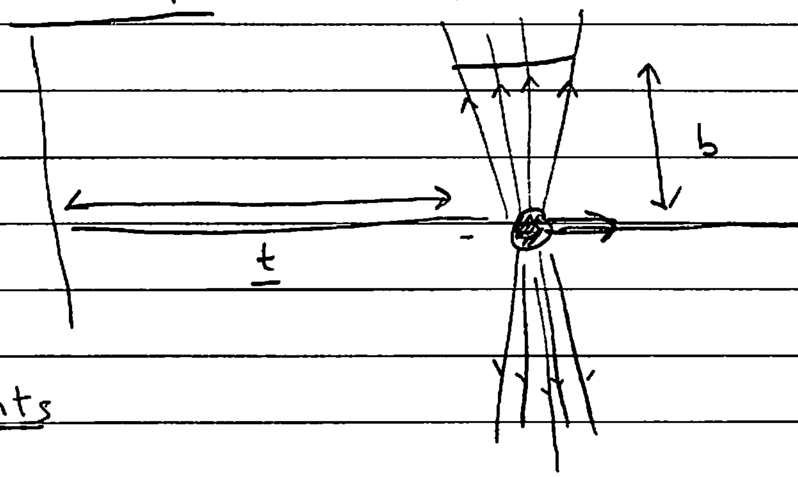


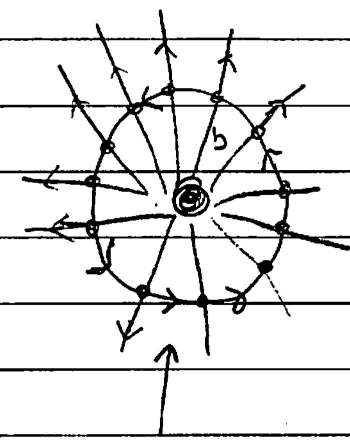
$$\underline{\vec{E}}_\perp(\underline{x}) = \frac{e\gamma\,\vec{b}}{(b^2 + \gamma^2(\underline{x} - v\underline{t})^2)^{3/2}} = \vec{E}_\perp(x)$$

$$\underline{\vec{B}} = \vec{\beta} \times \underline{\vec{E}}$$

So Basic Picture for γ large

Side View $\qquad$ $\underline{x} - v\underline{t} \sim b/\gamma$ $\qquad$ Head on View

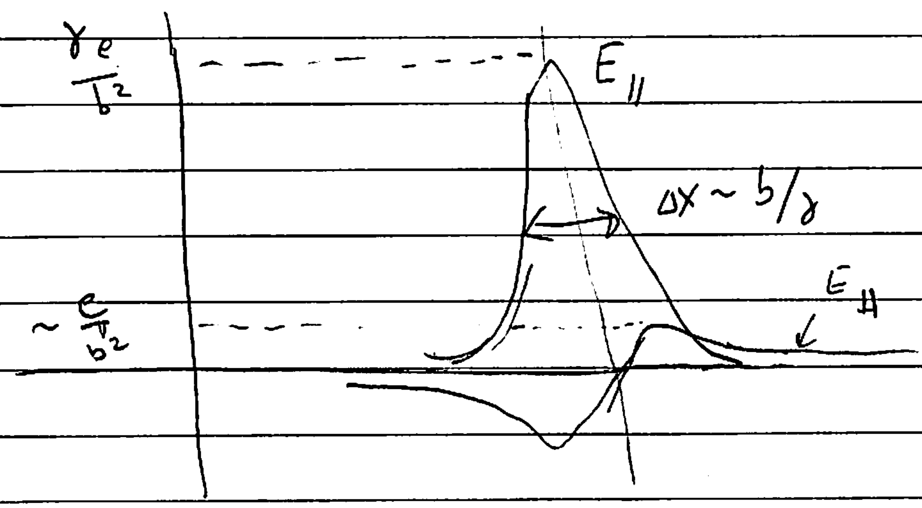




Comments

- For $\beta \to 1$ the $\vec{E}$ and $\vec{B}$ act almost like plane waves, i.e.
  $$|\vec{E}| \simeq |\vec{B}| \qquad \hat{E} \times \hat{B} \simeq \hat{z}$$
- B-field
- $\vec{S} = c\vec{E} \times \vec{B}$ points out of page

- Field Strengths:



- Find that the field gives a finite transverse kick (Homework). But no longitudinal kick.
  $$\Delta p_{\perp} = \frac{e^2}{2\pi b c}$$