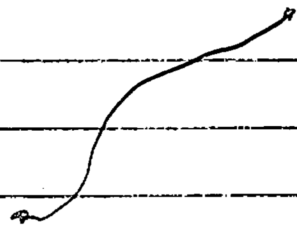
A primer on actions

- Todays lectur well deal with deriving EOM from actions



$$I_0 = \int dt \, \frac{1}{2} m \dot{x}^2 \qquad\qquad I_{int} = \int F_0 \, x \, dt$$

EOM:

$$\delta I_{TOT} = \delta I_0 + \delta I_{int}$$

$$= \int dt \, \delta x \left[-m\ddot{x} + F \right]$$

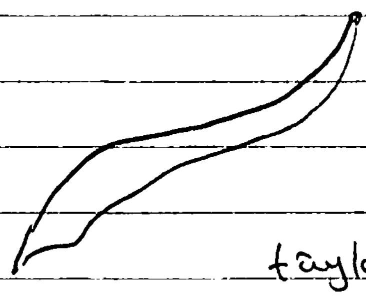
So EOM

$$\underbrace{m\ddot{x}} = \underbrace{F}$$

$$\underbrace{-\frac{\delta I_0}{\delta x}}_{\frac{dp}{dt}} = \underbrace{\frac{\delta I_{int}}{\delta x}}_{\text{force}}$$

Keep this in mind today!

Super Slow Discussion of variations



Vary the path

small

$$x(t) \to x(t) + \delta x(t)$$

vanishes at ends

taylor series

$$\frac{1}{2} m (\dot{x} + \delta\dot{x})^2 = \frac{1}{2} m \dot{x}^2 + m \dot{x}\,\delta\dot{x} + O(\delta x^2)$$

$$S \to S + \delta S = \int \frac{1}{2} m (\dot{x} + \delta\dot{x})^2 + F(x + \delta x)\; dt$$

$$= S + \int dt \left[m \dot{x}\,\delta\dot{x} + F\,\delta x \right] dt$$

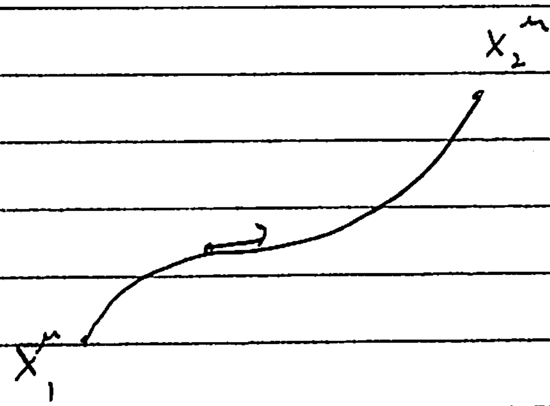
By parts

$$= S + \int dt \left[-\frac{d}{dt}(m\dot{x}) + F \right] \delta x(t)\; dt$$

This $\delta x(t)$ is arbitrary. The only way to guarranty that $\delta S = 0$ is if the thing in Square brackets vanishes.

Last Time

- Started to write down an action for the particles



$S = \int$ all possible forms which are lorentz invariant

parameter

Parametrize path by $x^\mu(\rho)$

$-\frac{dx^\mu}{d\rho}\frac{dx_\mu}{d\rho}$ invariant

Action should not depend on the parameterization

$$\frac{dx^\mu}{d\rho} \longrightarrow \frac{dx^\mu}{d\rho'}\frac{d\rho'}{d\rho}$$

This restricts the altion to:

$$I_0 = \underset{\text{const}}{k} \int dp \sqrt{-\frac{dx^\mu}{dp}\frac{dx_\mu}{dp}}$$

Often take p to be the propper time τ

Then ~~chose const to get non-rel action right.~~

$$I_0 = -\int mc^2 \, d\tau \qquad c\,d\tau = \sqrt{-dx^\mu dx_\mu}$$

So

$$= dp \sqrt{-\frac{dx^\mu}{dp}\frac{dx_\mu}{dp}}$$

$$d\tau = \frac{dt}{\gamma_u} = dt\sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

and

$$L = -mc^2 \left(1 - \frac{\dot{x}^2}{c^2}\right)^{1/2} \simeq -mc^2 + \frac{1}{2} m \dot{x}^2$$

Now $\quad \delta I_0 = -\delta \int dt \, mc^2 \sqrt{1 - \dot{x}^2/c^2}$

$$= +\int dt \; mc^2 \frac{1}{\cancel{2}\sqrt{1-\dot{x}^2/c^2}} \frac{+\cancel{2}\dot{x}^i \partial_t \delta x_i}{c^2}$$

$$\delta I_0 = -\int dt \; \delta x_i \left[\frac{d}{dt}\left(\gamma_u m u^i\right)\right]$$

So EOM is

$$\frac{d}{dt}(\gamma_u m \vec{u}) = 0 \implies$$ relativistic momentum is conserved

We would like to give a covariant treatment of these steps

Covariant Formulation

$$I = -\int d\tau \, mc^2 \qquad c\,d\tau = \sqrt{-dx^\mu dx_\mu}$$

Parametrize $x^\mu(p)$ $\quad c\,d\tau = \sqrt{-\frac{dx^\mu}{dp} \cdot \frac{dx_\mu}{dp}}\, dp$

So the variation $x^\mu \to x^\mu + \delta x^\mu(p)$

$$I = -\int \frac{dp}{c} \, mc^2 \left(-\frac{dx}{dp} \cdot \frac{dx}{dp}\right)^{1/2}$$

$$\delta I = -\int dp \; mc \frac{1}{\left(-\frac{dx}{dp} \cdot \frac{dx}{dp}\right)^{-1/2}} \, -\frac{dx^\mu}{dp} \cdot \frac{d\,\delta x_\mu}{dp}$$

integrate by parts

$$\delta I = -\int dp \left[\frac{d}{dp} \frac{mc}{\sqrt{-\dot{x}\cdot\dot{x}}} \frac{dx^\mu}{dp}\right] \delta x_\mu$$

Using $dp \frac{d}{dp} = d\tau \frac{d}{d\tau}$ and $\frac{d}{d\tau} = \frac{c}{\sqrt{-\frac{dx^\mu}{dp} \cdot \frac{dx_\mu}{dp}}} \frac{d}{dp}$

$$\boxed{\delta I = -\int d\tau \left[\frac{d}{d\tau}\left(m \frac{dx^\mu}{d\tau}\right)\right] \delta x_\mu}$$

The inteaction Lagrangian

$$I_{int} = \frac{e}{c}\int d\rho \, \frac{dx^{\mu}}{d\rho} A_{\mu}$$ ← invariant under reparentrization

constant e

only lorentz invariant linear in fields, $F_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$

$$= \frac{e}{c}\int d\tau \, \frac{dx^{\mu}}{d\tau} A_{\mu}$$

$$\boxed{I_{int} \simeq \int dt \left[\underbrace{-e\varphi}_{-PE} + \frac{d\vec{x}}{dt}\cdot\frac{\vec{A}}{c} \right]}$$

So

$$\delta I_{int} = \frac{e}{c}\int d\rho \left(\frac{d}{d\rho}\delta x^{\mu}\right) A_{\mu} + \frac{dx^{\mu}}{d\rho}\frac{\partial A_{\mu}}{\partial x^{\nu}}\delta x^{\nu}$$

$$= \frac{e}{c}\int d\rho \, \delta x^{\mu}\left(-\frac{dA_{\mu}}{d\rho}\right) + \frac{dx^{\mu}}{d\rho}\frac{\partial A_{\mu}}{\partial x^{\nu}}\delta x^{\nu}$$

$$\frac{dA_{\mu}}{d\rho} = \frac{\partial A_{\mu}}{\partial x^{\rho}}\frac{dx^{\rho}}{d\rho}$$

Relabelling indices

$$\delta I_{int} = \int d\rho \left[\frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha} \right] \frac{dx^\alpha}{d\rho} \delta x^\beta$$

$$\delta I_{int} = \int d\rho \; F_{\beta\alpha} \frac{dx^\alpha}{d\rho} \delta x^\beta$$

$$= \int d\tau \; F_{\beta\alpha} u^\alpha \delta x^\beta$$

Then

The term is [ ] must be zero.

$$\delta I_{TOT} = \delta I_0 + \delta I_{int}$$

$$\boxed{\delta I_{TOT} = \int d\tau \left[-m \frac{d^2 x_\beta}{d\tau^2} + F_{\beta\alpha} u^\alpha \right] \delta x^\beta}$$

Summary @ view to quantum mechanics

$$e^{iS[x]} = e^{ie\int d\tau \frac{dx^\mu}{d\tau} \frac{A_\mu}{c}} \qquad e^{i\frac{e}{c}\int dx_\mu A^\mu}$$

Wilson line

A Lagrangian for the fields

$$I[A] = \int \text{all possible lorentz invariants}$$
consistent with symmetries
and no more than: quadratic
in F

Given the field strength: $F^{\mu\nu}$ two possible forms:

- $-F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2) \leftarrow$ even under parity
- $F_{\mu\nu}\tilde{F}^{\mu\nu} = +2B\cdot E \leftarrow$ ~~odd under~~ parity

So

Lagrangian parity even

$$I[A] = \int d^4x \; a_1 F_{\mu\nu}F^{\mu\nu} + \cancel{a_2 F_{\mu\nu}\tilde{F}^{\mu\nu}}$$

$\hookrightarrow$ choose a_1 to be $-\frac{1}{4}$

- The $1/4$ is conventional

kinetic term

- The -1 guarrantees $(\partial_t \vec{A})^2 \sim E^2$ positive

The Action for the fields pg. 2

Then $\quad F^2 = F_{\mu\nu} F^{\mu\nu}$

$$I_{field} = \int d^4x \; \underbrace{-\frac{1}{4} F^2}$$

write $A_\mu \to A_\mu + \delta A_\mu$ and expand

see next pg.

$$\delta I_{field} = -\frac{1}{2}\int d^4x \; F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)$$

integrate by parts

$$= +\frac{1}{2}\int d^4x \; \partial_\mu F^{\mu\nu}\, \delta A_\nu - \partial_\nu F^{\mu\nu}\, \delta A_\mu$$

Relabel indices and use anti-symmetry of $F^{\mu\nu}$

$$= \int d^4x \; \delta A_\beta \left[\partial_\alpha F^{\alpha\beta} \right]$$

In general the field is coupled to currents

$$I_{int} = \int d^4x \; J^\mu \frac{A_\mu}{c}$$

For example for particle lagrangian

$$I_{int} = \int d\tau \; e \frac{dx^\mu}{d\tau} \frac{A_\mu}{c}$$

In general, <u>define</u> the current as

$$J_\mu \equiv \frac{\delta I_{int}}{\delta A_\mu} \quad \text{or} \quad \delta I_{int} = \int d^4x \; J^\mu \delta A_\mu$$

Slow motion variation of F^2

$$F^2 = F^{\mu\nu} F_{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

Now replace $A^\mu \to A^\mu + \delta A^\mu$ and expand in δA_μ

$$\begin{aligned} F^2 \to F^2 + \delta F^2 = F^2 &+ (\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu)(F_{\mu\nu}) \\ &+ (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) \\ = F^2 &+ (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F^{\mu\nu} \\ &+ F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) \\ = F^2 &+ 2 F^{\mu\nu} \delta F_{\mu\nu} \end{aligned}$$

So

$$\delta F^2 = 2F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)$$

The action of the field pg. 3

Then

$$\delta I_{field} + \delta I_{int} = \int d^4x \; \delta A_\beta \left[\partial_\alpha F^{\alpha\beta} + J^\beta \right]$$

And the field eqs are

$$\boxed{-\partial_\alpha F^{\alpha\beta} = J^\beta}$$

$$\underbrace{-\frac{\delta I_{field}}{\delta A_\beta}}_{m \cdot a} = \underbrace{\frac{\delta I_{int}}{\delta A_\beta}}_{\text{analogous to force}}$$

Gauge invariance & Current Conservation

$$\frac{\delta I_{int}}{\delta A_\mu(x)} \equiv J^\mu(x)$$

$$\delta I_{int} = \int J^\mu(x)\,\delta A_\mu(x)\, d^4x$$

Now suppose that under a gauge transformation I_{int} is unchanged:

$$I_{int} \xrightarrow[\text{gauge transform}]{} I'_{int} = I_{int}$$

i.e. $A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \Lambda(x)$

$\delta A'_\mu = \delta A_\mu + \partial_\mu \delta\Lambda$

or $\varphi \longrightarrow \varphi + \partial_t \Lambda$

$\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\Lambda$

Then

$$\delta I'_{int} - \delta I_{int} = 0 = \int J^\mu(x)\, \partial_\mu(\delta\Lambda)\, d^4x$$

Integration by parts gives

$$0 = -\int (\partial_\mu J^\mu)\, \delta\Lambda(x)\, d^4x$$

So since $\delta\Lambda$ is arbitrary we see that the gauge invariance of I_{int} implies current conservation

$$\boxed{\partial_\mu J^\mu = 0}$$

Last Time

- Discussed Action Principles

$$S = \underbrace{\int dt \, \frac{1}{2} m \dot{x}^2}_{S_0} + \underbrace{\int F x \, dt}_{S_{int}}$$

Find the extremum of the action $x(t) \to x + \delta x(t)$

$$\delta S = \int dt \; m \dot{x} \, \partial_t \delta x + F \delta x$$

$$\delta S = \int dt \; \delta x(t) \left[-m \ddot{x} + F \right] \quad \Rightarrow \quad -\frac{\delta S_0}{\delta x} = \frac{\delta S_{int}}{\delta x}$$

Then we went from there; $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$

$$S_0 = \underbrace{\int d\lambda \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}} \; mc}_{S_0} + \underbrace{e \frac{dx^\mu}{d\lambda} \frac{A_\mu}{c}}_{S_{int}}$$

$$\delta S = \int d\tau \; \delta x_\mu \left[\frac{-d}{d\tau} m \frac{dx^\mu}{d\tau} + \frac{e}{c} F^\mu{}_\nu \frac{dx^\nu}{d\tau} \right]$$

Gives motion of charged particles

$$S_{int} = e\int d\tau \, \frac{dx^\mu}{d\tau} \frac{A_\mu}{c}$$

$$\boxed{S_{int}^{nr} \simeq \int dt \; -e\varphi + \frac{\vec{v}}{c} \cdot \vec{A}}$$

This is how it will be used in non-rel quantum mechanics

Finally Discussed the action of E+M:

$$S_{TOT} = \underbrace{-\frac{1}{4}\int d^4x \; F_{\mu\nu} F^{\mu\nu}}_{S_0} + \underbrace{\int J_\mu \frac{A^\mu}{c} \, d^4x}_{S_{int}}$$

Now we look for extremum of $A_\mu \to A_\mu + \delta A_\mu$:

$$\delta S_0 = -\frac{1}{4}\delta \int d^4x \; (\partial_\mu A_\nu - \partial_\nu A_\mu)\, \eta^{\mu\mu'} \eta^{\nu\nu'} (\partial_{\mu'} A_{\nu'} - \partial_{\nu'} A_{\mu'})$$

$$= -\frac{1}{2}\int d^4x \; F^{\mu\nu} \underbrace{(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)}$$

$$\delta S_{TOT} = +\int d^4x \left[\partial_\mu F^{\mu\nu} + J^\nu\right] \delta A_\nu$$

$$J^\nu \equiv \frac{\delta S_{int}}{\delta A_\nu}$$

Today two items to clean up

- Gauge invariance and conservation laws
- Covariant Stress Tensor

Gauge - Invariance

Consider the interaction between the currents and the Maxwell field

$$I_{int} = \frac{1}{c} \int d^4x \; J^{\mu} A_{\mu}$$

$$\delta I_{int} = \frac{1}{c} \int d^4x \; J^{\mu} \, \delta A_{\mu}$$

Now make a gauge transformation

$$\delta A_{\mu} \longrightarrow \delta A_{\mu} + \frac{\partial \, \delta \Lambda}{\partial x^{\mu}}$$

If the action is gauge invariant

$$\delta I_{int} = \underbrace{\frac{1}{c} \int d^4x \; J^{\mu} \, \delta A_{\mu}}_{\delta I_{int}} + \frac{1}{c} \int d^4x \; J^{\mu} \frac{\partial \, \delta \Lambda}{\partial x^{\mu}}$$

$$0 = \frac{1}{c} \int d^4x \; J^{\mu} \frac{\partial \, \delta \Lambda}{\partial x^{\mu}}$$

$$0 = -\int d^4x \left(\frac{\partial J^{\mu}}{\partial x^{\mu}} \right) \delta A \quad \Longrightarrow \quad \boxed{\frac{\partial J^{\mu}}{\partial x^{\mu}} = 0}$$

Gauge invariance implies current conservation and vice-versa

Covariant Stress Tensor

- So far wrote the conservation laws non-covariantly

$$\frac{\partial u_{mech}}{\partial t} + \nabla \cdot \vec{S}_{mech} = \vec{j} \cdot \vec{E}$$

$$\frac{\partial g^{j}_{mech}}{\partial t} + \partial_i T^{ij}_{mech} = \rho \vec{E}^{j} + \left(\frac{\vec{j}}{c} \times \vec{B}\right)^{j}$$

Now divide the first eq by c, and slightly

$$\frac{1}{c}\frac{\partial u_{mech}}{\partial t} + \nabla \cdot \frac{S_{mech}}{c} = \frac{\vec{j}}{c} \cdot \vec{E} = F^{0}{}_{i} \frac{J^{i}}{c}$$

$$\frac{1}{c}\frac{\partial\, c g^{j}_{mech}}{\partial t} + \frac{\partial}{\partial x^{i}} T^{ij} = F^{i}{}_{0} \frac{J^{0}}{c} + F^{i}{}_{j} \frac{J^{j}}{c}$$

So the

$$\boxed{\frac{\partial}{\partial x^{\mu}} \Theta^{\mu\nu}_{mech} = \overbrace{F^{\nu}{}_{\rho} \frac{J^{\rho}}{c}}^{\text{compare to force law}}}$$

$$\frac{dp^{\mu}}{d\tau} = q F^{\mu}{}_{\nu} \frac{dx^{\nu}}{d\tau}$$

$$\Theta^{\mu\nu}_{mech} = \left(\begin{array}{c|c} u_{mech} & S_{mech}/c \\ \hline \frac{S_{mech}}{c} & T^{ij}_{mech} \end{array}\right)$$

$$\frac{S_{mech}}{c} = \Theta^{0i}$$

$$c g_{mech} = \frac{S_{mech}}{c} = \Theta^{i0}$$

Covariant Stress Tensor pg. 2

Then we showed

$$\frac{\vec{j}}{c} \cdot \vec{E} = -\left(\frac{1}{c} \frac{\partial u_{em}}{\partial t} + \nabla \cdot \frac{S_{em}}{c} \right)$$

and

recall $S/c = cg$

$$\left(\rho E + \frac{\vec{j}}{c} \times \vec{B} \right)^{j} = -\left(\frac{1}{c} \frac{\partial (\underbrace{c g_{em}})}{\partial t} + \frac{\partial}{\partial x^i} T^{ij}_{em} \right)$$

Can write this in covariant form

$$\boxed{F^{\nu}{}_{\rho} \frac{J^{\rho}}{c} = - \frac{\partial}{\partial x^{\mu}} \Theta^{\mu\nu}_{em}}$$

Where $\boxed{\Theta^{\mu\nu}_{em} = F^{\mu\lambda} F^{\nu}{}_{\lambda} + g^{\mu\nu} \frac{1}{4} F^2}$

$$\Theta^{\mu\nu}_{em} = \left(\begin{array}{c|c} u_{em} & \vec{S}/c \\ \hline \vec{S}/c & T^{ij} \\ & \end{array} \right)$$

Example

$$\Theta^{00} = \underbrace{F^{0i}}_{E^i} \underbrace{F^{0}{}_{i}}_{E^i} + \underbrace{g^{00}}_{-1} \underbrace{\left(\frac{-1}{4} F^2 \right)}_{\frac{1}{2}(E^2 - B^2)}$$

$$\Theta^{00} = \frac{1}{2} E^2 + \frac{1}{2} B^2$$

Covariant Stress pg. 3

So with that

$$\frac{\partial}{\partial x^\mu} \Theta^{\mu\nu}_{mech} = F^\nu{}_\rho \frac{J^\rho}{c}$$

$$\frac{\partial}{\partial x^\mu} \Theta^{\mu\nu}_{mech} = -\frac{\partial (\Theta^{\mu\nu}_{em})}{\partial x^\mu}$$

And thus

$$\frac{\partial}{\partial x^\mu} \left(\Theta^{\mu\nu}_{mech} + \Theta^{\mu\nu}_{em} \right) = 0$$

Covariant Prf that $F^{\nu}{}_{\rho} J^{\rho}/c = -\frac{\partial}{\partial x^{\mu}} \theta_{em}^{\mu\nu}$

$$F^{\nu}{}_{\rho} \frac{J^{\rho}}{c} = -F^{\nu}{}_{\rho} \frac{\partial F^{\mu\rho}}{\partial x^{\mu}}$$

$$= -\frac{\partial}{\partial x^{\mu}} (F^{\nu}{}_{\rho} F^{\mu\rho}) + F^{\mu\rho} \frac{\partial}{\partial x^{\mu}} F^{\nu}{}_{\rho}$$

Using the Jacobi-identity

$$F^{\mu\rho} \frac{\partial}{\partial x^{\mu}} F^{\nu}{}_{\rho} = g^{\nu\sigma} F^{\mu\rho} \partial_{\mu} F_{\sigma\rho}$$

So with the fact that multiplying by $F^{\mu\rho}$

$$\partial_{\mu} F_{\sigma\rho} + \partial_{\sigma} F_{\rho\mu} + \partial_{\rho} F_{\mu\sigma} = 0$$

$$\partial_{\mu} F_{\sigma\rho} - \partial_{\rho} F_{\sigma\mu} + \partial_{\sigma} F_{\rho\mu} = 0$$

Now

$$F^{\mu\rho} \partial_{\mu} F_{\sigma\rho} = \frac{1}{2} F^{\mu\rho} \left[\partial_{\mu} F_{\sigma\rho} - \partial_{\rho} F_{\sigma\mu} \right]$$

$$= \frac{1}{2} F^{\mu\rho} \left[-\partial_{\sigma} F_{\rho\mu} \right]$$

$$= \frac{1}{4} \frac{\partial}{\partial x^{\sigma}} F^{\mu\rho} F_{\mu\rho}$$

Covariant Prf pg. 2

So

$$F^{\nu}{}_{\rho} \frac{J^{\rho}}{c} = -F^{\nu}{}_{\rho} \frac{\partial}{\partial x^{\mu}} F^{\mu\rho}$$

$$= -\frac{\partial}{\partial x^{\mu}} F^{\nu}{}_{\rho} F^{\mu\rho} + \frac{\partial}{\partial x^{\sigma}} g^{\nu\sigma}\left(+\frac{1}{4} F^{2}\right)$$

$$F^{\nu}{}_{\rho} \frac{J^{\rho}}{c} = -\frac{\partial}{\partial x^{\mu}}\left[F^{\nu}{}_{\rho} F^{\mu\rho} + g^{\mu\nu}\left(-\frac{1}{4} F^{2}\right)\right]$$

$$= -\frac{\partial}{\partial x^{\mu}} \Theta^{\mu\nu}_{em}$$

$$\Theta^{\mu\nu}_{em} = F^{\mu\rho} F^{\nu}{}_{\rho} + g^{\mu\nu}\left(-\frac{1}{4} F^{2}\right)$$