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Charged Cylinder: \( \phi = 0 \) metal grounded plates

charged (insulating) cylindrical shell with charge per length \( \lambda \)

\( \phi = 0 \)

\( R \)

L

Determine \( \phi (r, z) \) both inside and outside the cylinder. Concentrate on \( z = 0 \)

Warm up questions:

1. What are the dimensionfull parameters?

2. What are the boundary conditions?
   - what is the perpendicular directions
   - what are the parallel directions

3. What do you expect when \( L \gg R \) at \( z = 0 \)
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Solution: (Qualitative)

1. \( \lambda, L, R, \) and \( z, \rho \)

at \( z = 0 \) the solution depends on how \( \rho \) compares to \( L, R \)

2. Boundary conditions:

\[ \Phi = 0 \text{ at } z = \pm L, \]

\[ \left. \frac{E_\rho}{R+\epsilon} \right|_{R-\epsilon} = \sigma = \frac{\lambda}{2\pi R} \text{ for all } z \]

\( \epsilon \) take \( z, \phi \) to be parallel directions and \( \rho \) to be perp directions

3. For

\( R \ll \rho \ll L \) the walls are infinitely far away, Gauss law gives:

\[ E_\rho = \frac{\lambda}{2\pi \rho} \]

\[ \Phi = -\frac{\lambda}{2\pi} \log (\rho) + \text{Const} \]
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Then

- As \( \rho \sim L \) start to feel the effect of walls

- for \( R \ll L \ll \rho \) the fields will decrease exponentially

- Inside the cylinder expect

\[
\phi = \text{const up to corrections suppressed by } \frac{R^2}{L^2}
\]

\[
\phi(\rho) \big|_{\rho=L} = 0
\]

\[
\phi = -\frac{\lambda}{2\pi} \ln \rho + \text{Const}
\]

\[
e^{-\frac{\lambda}{2\pi} \rho}
\]
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Solution (Quantitative)

\[ \psi = \sum_k R_k(p) Z_k(z) \]

The Laplace Eqn. inside and outside

\[ -\nabla^2 \psi = 0 \]

\[ \begin{bmatrix} -\frac{1}{p^2} \frac{\partial^2 \psi}{\partial p^2} + \frac{\partial^2 \psi}{\partial p \partial z} + \frac{\partial^2 \psi}{\partial z^2} \end{bmatrix} \psi = 0 \]

Leads to the separated eqns:

\[ \begin{bmatrix} -\frac{1}{p^2} \frac{\partial^2 \psi}{\partial p^2} + k^2 \end{bmatrix} R_k(p) = 0 \]

\[ \begin{bmatrix} -\frac{\partial^2 \psi}{\partial z^2} - k^2 \end{bmatrix} Z_k(z) = 0 \]

\( Z_k = A_k^o \cos k_0 z + \sum_k A_k \cos k z + B_k \sin k z \)

\( k \neq 0 \)

General solution

Solution

Boundary conditions at \( z = -L/2 \) and \( z = L/2 \)

+ symmetry \( \psi(z = -L/2) = \psi(z = L/2) = 0 \)

\[ Z_k = \sum \Lambda_k \cos \left( \frac{k_n \pi}{L} \right) \]
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Now since the function must vanish at \( z = -L/2 \) and \( L/2 \)

\[ k_n = \frac{(2n+1)\pi}{L} \]

\[ n = 0 \quad n = 1 \quad \ldots \]
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From the radial direction

\[ R_k(p) = A_k \, I_0(kp) + B_k \, K_0(kp) \]

Asymptotics:

- For \( x \ll 1 \):
  \[ I_0(x) \Rightarrow 1 + x^2/4 + \ldots \]
  \[ K_0(x) \Rightarrow -[\log x + \gamma_E] I_0 + o(x) \]

- For \( x \gg 1 \):
  \[ I_0 = \frac{e^x}{\sqrt{2\pi x}} (1 + o(1/x)) \]
  \[ K_0 = \frac{e^{-x}}{\sqrt{2\pi x}} \]

So inside the cylinder

\[ R_k(p) = A_k \, I_0(kp) \]

And outside

\[ R_k(p) = B_k \, K_0(kp) \]

Continuity at \( p = R \) shows \( B_k = I_0(kR) \) \( A_k = K_0(kR) \)

\[ R_k(p) = C_k \left[ K_0(kR) I_0(kp) \Theta(p-R) + I_0(kR) K_0(kp) \Theta(p-R) \right] \]
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So the solution at this point is

\[ \psi(p, z) = \sum \frac{C_n}{n} \left[ I_0(k_p) K_0(k_p) \Theta(p - R) \right. \\
\left. + I_0(k_R) K_0(k_p) \Theta(p - R) \right]\]

\( \times \cos(k_n z) \)

where \( k_n = \frac{(2n+1)\pi}{L} \quad n=0, 1, 2, 3, \ldots \).

From the jump condition can determine \( C_n \):

\[ \frac{E^\text{out}}{2}\pi R - \frac{E^\text{in}}{2}\pi R = \lambda \]

\[ E^\text{out} = -\sum C_n K_0(k_R) I_0'(k_R) k_n \cos(k_n z) \]

\[ E^\text{in} = -\sum C_n I_0(k_R) K_0'(k_R) k_n \cos(k_n z) \]

\[ E_p(k) = \frac{2}{L} \int_{-L/2}^{L/2} \cos(k_n z) E^\text{out}_p \]

\[ = -C_n K_0(k_R) I_0'(k_R) k_n \]

\[ E^\text{in}_p(k) = -C_n I_0(k_R) K_0'(k_R) k_n \]
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Now

$$E^{\text{out}}(k) - E^{\text{in}}(k) = \frac{1}{L} \int_{-L/2}^{L/2} \frac{A}{2\pi R} \cos(k_n z) = \frac{\lambda_0}{2\pi}$$

$$= \frac{(-1)^n 2\lambda}{(1 + 2n)\pi^2 R}$$

So we get that

$$\text{Wronskian}$$

$$- C_n \left[ I_0(k_n R) K'_0(k_n R) - K_0(k_n R) I'_0(k_n R) \right] k_n$$

$$= \frac{(-1)^n 2\lambda}{(1 + 2n)\pi^2 R}$$

Recognizing the Wronskian (which often appears in jump conditions, and we know that this will take a simple form. From Bessel's eqn:

$$\left[ -\frac{1}{2} \frac{\partial^2}{\partial r^2} + k^2 \right] (I_0(kr) \text{ or } K_0(kr)) = 0$$

$$p(x) \text{ of Sturm-Liouville eqn} \Rightarrow p(x) = 1$$

We know that \( p \times \text{Wronsk} (kr) = \text{const} \)

Use series to show \( \text{Wronsk} \bigg|_{k=\infty} = \frac{-1}{k\pi} \bigg|_{p=\infty} = \frac{-1}{k\pi} \)
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So

\[- C_n \begin{bmatrix} -1 \\ K_{pR} \end{bmatrix} = (-1)^n \frac{2\lambda}{(1+2n)\pi^2R} \]

So \( C_n \) is

\[ C_n = (-1)^n \frac{2\lambda}{(1+2n)\pi^2} \]

Thus we have determined the full solution

\[ \psi(p, z) = \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2} \times \left[ K_n(kR) I_0(kp) \Theta(R-p) + I_0(kR) K_0(kp) \Theta(p-R) \right] \times \cos(k_n z) \]

Let's look at \( z=0 \) and outside

\[ \psi(p) = \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2} I_0(kR) K_0(kp) \Theta(p-R) \]
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for \( p > R \) but \( p \ll L \) then

\[ k_n p \approx \frac{(2n+1)\pi p}{L} \ll 1 \quad \text{for almost all } n \]

and

\[ I_0 = 1, \quad k_0 \approx -\ln k_n p - \frac{2-\delta}{2} - \ln k_n + \ln k_n \]

\[ \Psi(p) = \sum_{n=0}^{\infty} (-1)^n \frac{2}{(1+2n)\pi^2} \left( -\ln p + \ln k_n + \text{const} \right) \]

\[ \Psi(p) = -\frac{1}{2\pi} \ln p + \text{const} \]

we used that

\[ \sum_{n=0}^{\infty} \frac{2}{(1+2n)\pi^2} = \frac{1}{2} \]

So

\[ \Psi(p) = -\frac{1}{2\pi} \ln p + \text{const} \]

\[ \Psi(p) = -\frac{1}{2\pi} \ln p + \text{const} \]

\[ K_0(k_n p) \approx \frac{1}{\sqrt{2\pi k_n p}} e^{-k_n p} \quad k_n = \frac{(2n+1)\pi p}{L} \]

for \( p \) large \( R \ll L \ll p \) find

\[ K_0(k_n p) \ll \frac{1}{\sqrt{2\pi k_n p}} e^{-k_n p} \]

The larger the \( n \) the more it is suppressed. Keep only the \( n=0 \) term
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Find outside

\[ \psi (p) = \sum_{n=0}^{\infty} \frac{(-1)^n 2 I_0(kR) K_0(kp)}{(1 + 2n) \pi^2} \]

\[ \psi (p) \approx \frac{1}{\pi^2} e^{-\frac{\pi p}{L}} \sqrt{\frac{\pi}{2(\pi p/L)}} \quad \text{only} \]

\[ \psi (p) = \frac{\sqrt{2}}{\pi^2} e^{-\frac{\pi p}{L}} \sqrt{\frac{\pi}{(p/L)}} \]

+ asympt
L/R = 10

\[ \frac{\varphi(\rho)}{\gamma} \]

not exactly flat = flat + \( O(\rho^2/L) \)

two terms of series

\[ \frac{\lambda \sqrt{2}}{\pi^2} \frac{e^{-\pi \rho/L}}{\sqrt{\rho/L}} \]

\[ -\frac{\lambda \log \rho + \text{Const}}{2\pi} \]