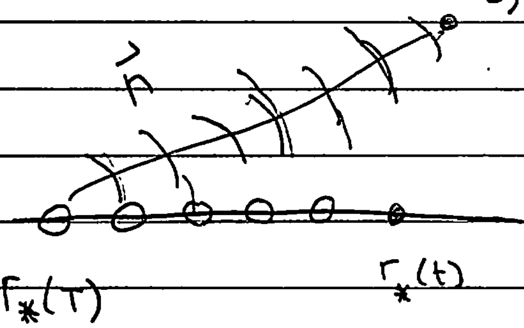
Last Time

Started to discuss radiation from moving charge

t, r = observation point



$$\vec{n} \equiv \frac{\vec{r} - r_*(T)}{|\vec{r} - r_*(T)|}$$

The fields at the observation point reflect the particle's past motion at time T

$$c(t - T) = |\vec{r} - \vec{r}_*(T)|$$

$$T = t - \frac{|\vec{r} - \vec{r}_*(T)|}{c} \qquad (\text{eq. } \star)$$

Thus

$T \equiv$ the retarded time is a function of t, r which is found by eq $\star$

$= T(t, r)$

We found the fields by solving the wave eqn

$$\left.\begin{aligned} -\Box\varphi &= \rho \\ -\Box\vec{A} &= J/c \end{aligned}\right\} \qquad \begin{aligned} \rho &= e\,\delta^3(\vec{r}_0 - r_*(t_0)) \\ \vec{J} &= e\,V(t_0)\,\delta^3(\vec{r}_0 - \vec{r}_*(t_0)) \end{aligned}$$

Then

$$\varphi(t,r) = \int d^3r_0\,dt_0\,\frac{\delta(t - t_0 - |r - r_0|/c)}{4\pi|\vec{r} - \vec{r}_0|}\,e\,\delta^3(\vec{r}_0 - r_*(t_0))$$

$$= \int dt_0\,\delta(t - t_0 - |r - r_*(t_0)|)\,\frac{e}{4\pi|\vec{r} - r_*(t_0)|}$$

Doing the t_0 integral, picking up a Jacobian

$$\boxed{\begin{aligned} \varphi(t,r) &= \frac{e}{4\pi|\vec{r} - r_*(T)|}\,\frac{1}{\left(1 - \vec{n}\cdot\frac{\vec{V}}{c}(T)\right)} \\ \vec{A}(t,\vec{r}) &= \frac{e\,\vec{V}(T)}{4\pi|\vec{r} - r_*(T)|}\,\frac{1}{\left(1 - \vec{n}\cdot\frac{\vec{V}}{c}(T)\right)} \end{aligned}}$$

Lienard Wiechert

$$|\vec{r} - r_*(T)| \equiv R(T)$$

Can be written covariantly

$$\underbrace{c(t-T)}_{\Delta T} = \underbrace{|r - r_*(T)|}_{\Delta R} \qquad \Delta X^\mu = (c\Delta T, \Delta R)$$

Leading to the retarded condition

$(\Delta X)^2 = 0 \leftarrow$ the two points are light-like separated

Then

$$A^\mu = -\frac{e\, U^\mu(T)}{4\pi\, U\cdot\Delta X} = -\frac{e\, V^\mu}{4\pi\, V\cdot\Delta X} \qquad V^\mu = (1, \vec{v})$$

$$U^\mu = (\gamma, \gamma v)$$

Excercise (Important!)

$$T = t - \frac{|\vec{r} - r_*(T)|}{c}$$

Show that

$$\frac{\partial T}{\partial t} = \frac{1}{1 - \vec{n} \cdot \vec{v}/c} \leftarrow \text{graded}$$

$$\frac{\partial T}{\partial r^i} = \frac{-n_i/c}{(1 - n \cdot v/c)} \qquad \text{(Extra credit!)}$$

Interpret $\frac{\partial T}{\partial t}$ physically! (Super-extra-credit)

Solution

- $$\frac{\partial T}{\partial t} = 1 - \frac{\partial}{\partial t}\frac{|r - r_*(T)|}{c}$$

$$\frac{\partial T}{\partial t} = 1 + \frac{n \cdot v}{c}\frac{\partial T}{\partial t} \qquad -\frac{(\vec{r} - \vec{r}_*)}{|r - r_*|} \cdot d\vec{r}_* = d|\vec{r} - \vec{r}_*|$$

$$\boxed{\frac{\partial T}{\partial t} = \frac{1}{1 - n \cdot v/c}}$$

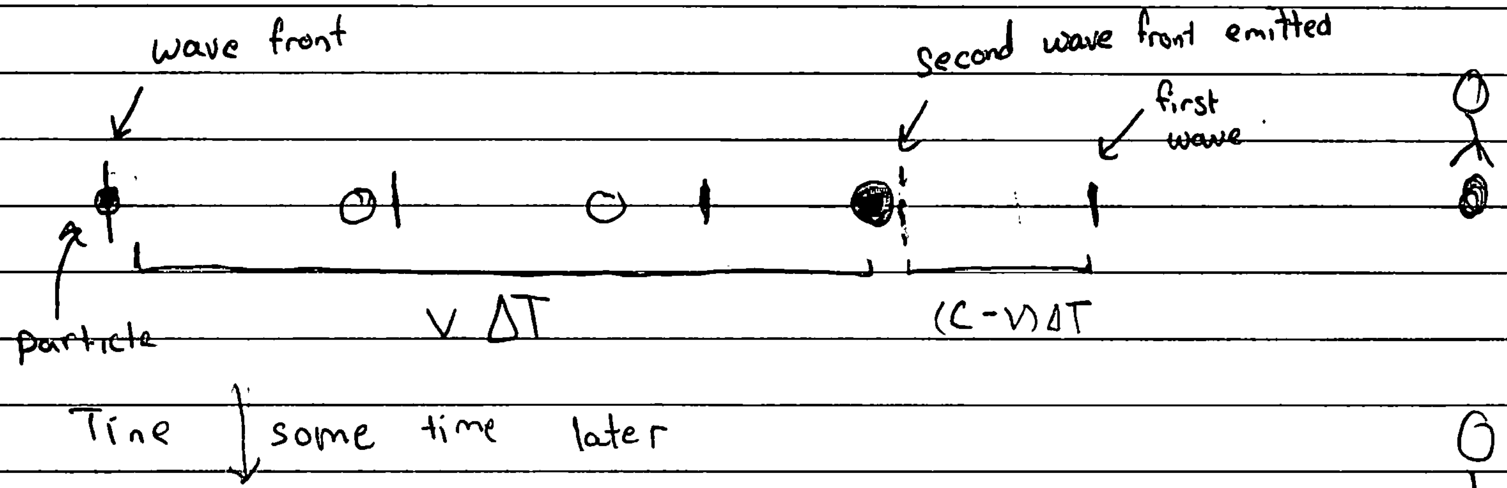
- Similarly

$$\frac{\partial T}{\partial r^i} = -\frac{(r - r_*)^j}{|r - r_*|} \cdot \left(\delta_{ji} - v_j \frac{\partial T}{\partial r^i}\right)$$

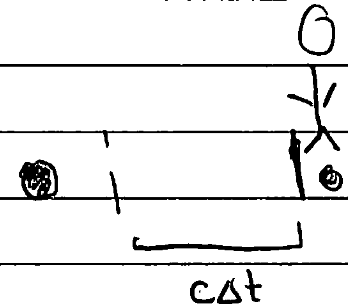
$$\frac{\partial T}{\partial r^i} = -n_i + (n \cdot v)\frac{\partial T}{\partial r^i}$$

$$\frac{\partial T}{\partial r^i} = \frac{-n_i/c}{\left(1 - \frac{n \cdot v}{c}\right)}$$

Physical Interpretation of "collinear factor" $= \dfrac{\partial T}{\partial t} = \dfrac{1}{(1 - n \cdot v/c)}$



So



$$c\Delta t = (c - v)\Delta T$$

$$\frac{1}{1 - \frac{v}{c}} = \frac{\Delta T}{\Delta t}$$

- So we see that $\dfrac{\Delta T}{\Delta t}$ goes to infinity as $\dfrac{v}{c} \to 1$, because particle is keeping up @ radiation.

So now lets evaluate the far field

$$\varphi = \frac{e}{4\pi R} \frac{1}{(1 - \vec{n}\cdot\vec{v}/c)}$$

$$\vec{A} = \frac{e}{4\pi R} \frac{\vec{v}/c}{(1 - \vec{n}\cdot\vec{v}/c)}$$

with

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\varphi$$

Setting $c=1$. Never differentiating $\vec{n}$ or $1/R$ in the far field limit

$$-\frac{\partial A^j}{\partial t} = \frac{e}{4\pi R}\left[\frac{-v^j}{(1-n\cdot v)^2}(n\cdot a)\frac{\partial T}{\partial t} + \frac{-a^j}{(1-n\cdot v)}\frac{\partial T}{\partial t}\right]$$

$$-\frac{\partial A^j}{\partial t} = \frac{e}{4\pi R}\frac{-v^j(n\cdot a) - a^j(1-n\cdot v)}{(1-n\cdot v)^3}$$

$$-\partial^j\varphi = \frac{e}{4\pi R}\frac{+1}{(1-n\cdot v)^3}\, n\cdot a\, n^j$$

$$\vec{E} = \frac{e}{4\pi R}\frac{1}{(1-n\cdot v)^3}\left[\underbrace{(-a^j + n^j(n\cdot a))}_{n\times(n\times a)} + \underbrace{(a^j(n\cdot v) - v^j(n\cdot a))}_{n\times(-\vec{v}\times a)}\right]$$

Find (restoring c)

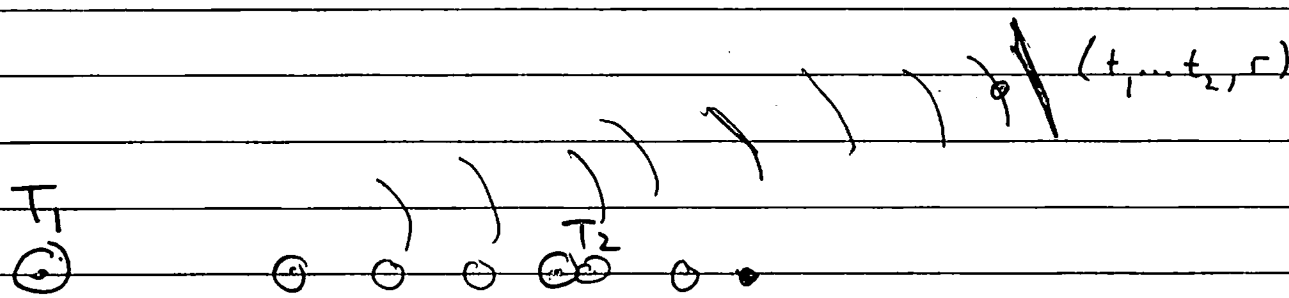
$$\boxed{\begin{aligned}\vec{E} &= \frac{e}{4\pi R c} \frac{n \times ((\vec{n}-\vec{\beta}) \times \dot{\vec{\beta}})}{(1-\beta\cdot n)^3} \\ B &= n \times \vec{E}\end{aligned}}$$

← Far fields of Lienard Wiechert potentials

So now that we know the fields Lets calculate the radiated power.

- We want to know the energy radiated during a finite period of acceleration from $T_1 \ldots T_2$

The energy per area at (t, r)



$$E = \int_{t_1 = T_1 + R(T_1)/c}^{t_2 = T_2 + R(T_2)/c} dt\, \vec{S}\cdot\vec{n} = \int_{T_1}^{T_2} \frac{dt}{dT} (S\cdot n)\, dT$$

So we see that the physically relevant quantity is

$$\frac{dP(T)}{d\Omega} = \text{energy per } \Omega \text{ per } dT$$

$$= R^2 (S \cdot n) \frac{dt}{dT}$$

$$= R^2 (\vec{S} \cdot \vec{n}) (1 - \beta \cdot n)$$

Given the fields we have

$$\boxed{\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c} \frac{|\vec{n} \times ((\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}})|^2}{(1 - \vec{n} \cdot \beta)^5}}$$

We used

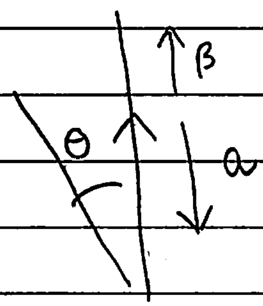
$$\vec{S} \cdot \vec{n} = c\,|E|^2$$

Then lets take the simplest case

$$\frac{a}{c} = \dot{\vec{\beta}} \ // \ \vec{\beta}$$

Then

$$\left| n \times n \times \frac{\vec{a}}{c} \right| = \bar{a}_{\perp} = a \sin\theta$$



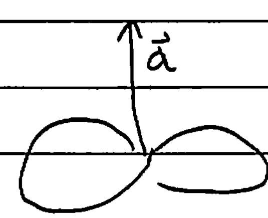
And then

$$\boxed{\frac{dP(T)}{d\Omega} = \frac{e^2 a^2}{16\pi^2 c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}}$$

Comments

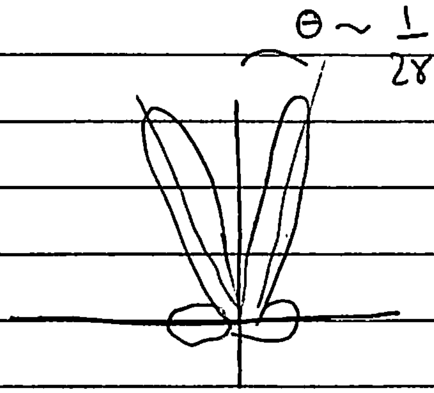
① When $\beta \to 0$ get the old result (non-rel)

$$\frac{dP}{d\Omega} = \frac{e^2 a^2}{16\pi^2 c^3} \sin^2\theta$$



② As $\beta \to 1$, the factors $(1-\beta\cos\theta) \to 0$ as $\theta \to 0$. The radiation becomes very collinear with the direction of motion

For $\gamma \simeq 2$

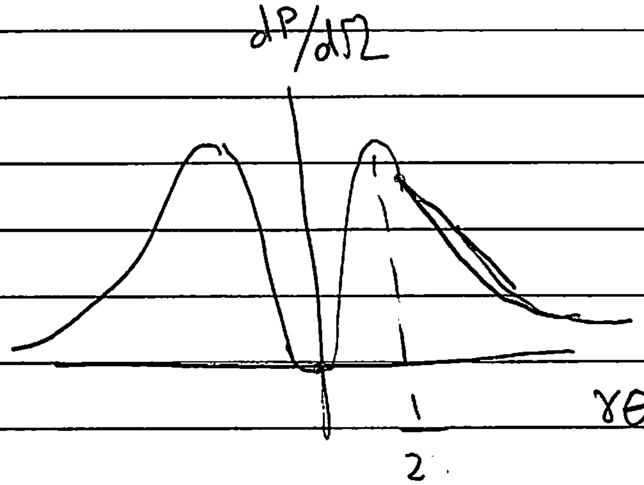


For $\beta \to 1$, $\gamma \to \frac{1}{\sqrt{(1-\beta)2}}$, $\cos\theta \simeq 1 - \frac{\theta^2}{2}$, $\sin\theta \simeq \theta$

Find

$$\frac{dP}{d\Omega} = \frac{e^2}{2\pi^2} \frac{a^2}{c^3} \gamma^8 \frac{(\gamma\theta)^2}{[1 + (\gamma\theta)^2]^5}$$

γ can be very large, e.g. a 50 GeV electron has $\gamma = 10^5$.



So

$$\theta_{max} = \frac{1}{2\gamma}$$

$$\theta_{max} = \frac{mc^2}{2E}$$