Last Time

- Discussed the radiation from charged particles moving with $\gamma \approx 10^5$.

![Diagram of time between emission and arrival events $\Delta t$.

Kinematics give

$$\frac{\Delta t}{\Delta T} = \frac{1}{1 - n \cdot v(T)}$$

$T(t, r) = \text{retarded time}$

$$T = t - 1 - \frac{r^2 - r_0(T)}{c}$$

Worked out the fields from a charged particle

$$-\Box A^\mu = \frac{J^\mu}{c}$$

by solving wave eqn.
Last Time (Continued)

Then

\[ \Psi = \frac{e}{4\pi R \left( 1 - \n \cdot \mathbf{V}(T) \right)} \quad \tilde{A} = e \frac{\mathbf{V}(T)}{4\pi R \left( 1 - \n \cdot \mathbf{V}(T) \right)} \]

\[ R = \left| \mathbf{r} - \mathbf{r}_*(T) \right| \quad \tilde{n} = \frac{\mathbf{r} - \mathbf{r}_*(T)}{\left| \mathbf{r} - \mathbf{r}_*(T) \right|} \]

So we can also write this covariantly (Aside)

\[ c(t-T) = \left| \mathbf{r} - \mathbf{r}_*(T) \right| \quad \Delta X^\mu = (\Delta X^0, \Delta \mathbf{X}) \]

\[ = \Delta X^0 \quad \Delta \mathbf{X} \]

Then the retarded condition says

\[ (\Delta X)^2 = 0 \] — the observation point and emission points are lightlike separated

Then

\[ A^\mu = -e \frac{V^\mu}{4\pi \mathbf{V} \cdot \Delta X} \]

\[ = -e \frac{U^\mu}{4\pi \mathbf{U} \cdot \Delta X} \] (end Aside)
Last Time (Continued)

Then once we have the potentials we find the fields

\[ E = \frac{e^2}{4\pi \mu_0 c^2} \frac{n \times (n - \beta) \times \dot{a}}{(1 - n \cdot \beta)^3} \]

\[ \vec{B} = n \times \vec{E} \]

Once we know the fields we can ask for the energy received per time per solid angle

\[ \frac{dP(t)}{dS} = \frac{dW}{dt \, d\Omega} \]

\[ = \frac{1}{S \cdot \pi R^2} \]

\[ = \frac{e^2}{16\pi^2 \mu_0 c^3} \left[ \frac{|n \times (n - \beta) \times \dot{a}|^2}{(1 - n \cdot \beta)^6} \right] \]

all quantities \( \vec{E} \) evaluated \( \text{set out at } T \)

Square of electric field

This is the right quantity to calculate if you want to know if the detector burns up in given time interval
However, you might want to know how much energy was radiated during a certain period of acceleration from $T_1$ to $T_2$.

\[
\frac{dP(T)}{d\Omega} = \frac{dW}{d\Omega d\tau d\omega} \frac{dW}{d\tau d\omega dt} \\
= \frac{e^2}{16\pi^2 c^3} \frac{1}{(1 - n \cdot \beta)^5} \left( n \times (n - \beta) \times \hat{a}\right)^2
\]

**Velocity and Acceleration Parallel**

\[
\frac{dP(t)}{d\Omega} = \frac{e^2}{16\pi^2 c^3} \frac{a_t^2}{(1 - n \cdot \beta)^5} \hat{a} = -\hat{n} \times (\hat{n} \times \hat{a})
\]

So taking $\beta$ and $\alpha$ on z-axis

\[
\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c^3} \frac{a^2 \sin^2 \theta}{(1 - \beta \cos \theta)^5}
\]
Comments about Eq. (velocity // to Accel)

1) Then we saw the Larmor formula for $\beta \to 0$

$$\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c^3} \frac{a^2 \sin^2 \theta}{\beta \cos \theta}$$

2) And started to think about the relativistic limit: $\beta \approx 0.99999$

$$\frac{dP}{d\Omega} = \frac{e^2}{16\pi^2 c^3 \left(1 - \beta \cos \theta \right)^5}$$

For $\beta = 1$ this collinear factor $\to 0$

Use:

$$\gamma = \frac{1}{\sqrt{(1-\beta)(1+\beta)}} = \frac{1}{\sqrt{2(1-\beta)}}$$

Expand

$$\frac{1}{(1-\beta \cos \theta)} = \frac{2 \gamma^2 (1 + (\gamma \theta)^2)}{(1 + (\gamma \theta)^2)^5}$$

And find that Eq. **

$$\frac{dP(T)}{d\Omega} = \frac{2}{\pi} \frac{e^2 a^2}{c^3} \gamma^8 \left( \frac{(\gamma \theta)^2}{(1 + (\gamma \theta)^2)^5} \right)$$
Comments about Eq. (continued) (velocity // accel)

So find:

\[ \frac{dP}{d\Omega} \]

\[ \frac{1}{2} \gamma \theta \]

The "dead-cone" from this picture:

\[ \theta \text{ typical} \sim \frac{1}{\gamma} \sim \frac{mc^2}{E} \]

Typical picture in heavy quark jets

For electron: \( mc^2 \sim 0.5 \text{ MeV} \)

\[ \theta \text{ typical} \sim \frac{1}{20,000} \sim \frac{1}{100}^{\circ} \]
Power when velocity & accel are orthogonal

Now let's return to the more general expression

\[
\frac{dP(T)}{d\Sigma} = \frac{dW}{dT \, d\Sigma} = \frac{e^2}{16 \pi^2 c^3} \left\| \mathbf{n} \times (\mathbf{v} - \mathbf{\beta}) \times \mathbf{a} \right\|^2 \frac{1}{(1 - \mathbf{n} \cdot \mathbf{\beta})^5}
\]

We so far considered parallel motion and acceleration. Then in this case

\[ \beta \]

\[ \alpha \]

Now we want to consider circular motion

Find a similar pattern of enhanced collinear radiation:

\[
\frac{dP(T)}{d\Sigma} \propto \frac{e^2}{c^3} \frac{a^2 \gamma^6}{(1 + (\gamma \theta)^2)^3 (1 + (\gamma \theta)^2)^5} \left[ \frac{1}{4(\gamma \theta)^2 \cos^2 \phi} \right]
\]
Total Power - Brute force Method

Now let's calculate the total power radiated

\[ P(T) = \frac{dW}{dT} = \int d\Omega \frac{e^2}{16\pi^2 c^3} \frac{\ln x (\hat{n} \cdot \beta) \times \hat{a}}{(1 - \hat{n} \cdot \beta)^5} \]

* Brute force: (not too important)

* this a rotationally invariant integral use:

\[ \hat{\beta} = (0, 0, \hat{\nu}) \]

\[ \hat{\alpha} = (a_\perp, 0, a_\parallel) = \hat{a}_\parallel + \hat{a}_\perp \]

\[ \hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi) \]

\[ \hat{n} \times \hat{n} \times \hat{a} = -\hat{a}_\perp \]

\[ -\hat{n} \times \hat{\beta} \times \hat{a} = -(\hat{n} \cdot \hat{a}) \hat{\beta} + (\hat{n} \cdot \hat{\beta}) (\hat{a}_\parallel + \hat{a}_\perp) \]

\[ S_0 = -(1 - \hat{n} \cdot \hat{\beta}) \hat{a}_\perp + [(\hat{n} \cdot \hat{\beta}) \hat{a}_\parallel - (\hat{n} \cdot \hat{a}) \hat{\beta}] = \hat{n} \times (\hat{n} - \hat{\beta}) \times \hat{a} \]

Plug in, do all integrals in \( \frac{\hat{a}^2}{R^2} \), take a deep breath...

\[ P = \frac{e^2}{4\pi} \frac{2}{3c^3} \left[ \frac{a_\parallel^2 + a_\perp^2}{\hat{a}^2} \right] \]

Lienard - Wiechert 1898 very important
Analysis of Total Power Radiated

This week homework, use

$$A^\mu = \frac{d^2 \chi^\mu}{dt^2}$$

In LRF of particle \((LRF = \text{Local rest frame})\)

$$A^\mu = \begin{pmatrix} 0 \\ \alpha_{11} \\ \alpha_{1} \end{pmatrix} \quad A^\mu A_\mu = \alpha_{11}^2 + \alpha_{1}^2$$

Show that:

$$\alpha_{11} = \frac{\alpha_{11}}{8} \quad \alpha_{1} = \frac{\alpha_{1}}{8}$$

see solution at end of lecture

So

$$\gamma^6 \left[ \frac{\alpha_{11}^2 + \alpha_{1}^2}{8^2} \right] = \alpha_{11}^2 + \alpha_{1}^2 = A^\mu A_\mu$$

And

$$P = \frac{e^2}{4\pi} \frac{2}{3c^3} A^\mu A_\mu \quad \text{relativistic generalization of Larmour}$$

$$P = \frac{e^2}{4\pi} \frac{2}{3mc^3} \frac{dp^\mu}{dt} \frac{dp_\mu}{dt}$$
Total Power  - (Pure Thinking)

Could Perhaps Guess This Result

Rest Frame of Particle

\[ \Delta E = \frac{e^2}{4\pi} \frac{2}{3c^3} \int \Delta \dot{a} \Delta t \]

\[ \Delta \vec{p} = 0 \quad \text{since radiate symmetrically and transversally} \]

\[ \Delta t = \Delta t \]

\[ \Delta x = 0 \]

So in any other frame

\[ \Delta E = \gamma \Delta E \]

\[ \Delta t = \gamma \Delta t \]

And

\[ \text{total power} = \frac{\Delta E}{\Delta t} = \frac{\Delta E}{\Delta t} = \text{invariant under boost} \]

\[ = \frac{e^2}{4\pi} \frac{2}{3c^3} \int A^m A_m \]

true in rest frame

true in all
Linear vs. Circular Accelerators

In general the energy radiated is

\[ P(T) = \frac{e^2}{4\pi^2} \frac{2}{3 m^2 c^3} \frac{\text{d}p^2}{\text{d}t} \]

for a linear accelerator, \( \frac{\text{d}p}{\text{d}t} \) is parallel to the motion

\[ \frac{\text{d}p^2}{\text{d}t} = \frac{\text{d}p}{\text{d}t}, \quad \frac{\text{d}p^0}{\text{d}t} = \frac{\text{d}E^0}{\text{d}t} \]

\[ E = \frac{\sqrt{p^2 + (mc^2)^2}}{c^2} \]

\[ = \frac{\gamma}{c} \frac{\text{d}p}{\text{d}t} \]

\[ = \left( \frac{\gamma}{c} \right) \cdot \gamma \frac{\text{d}p}{\text{d}t} \]

So

\[ P(T) = \frac{e^2}{4\pi^2} \frac{2}{3 m^2 c^3} \frac{(\text{d}p^2)^2}{\text{d}t} \]

- used \( 1 - \left( \frac{v}{c} \right)^2 = \frac{1}{\gamma^2} \)

\( \gamma \) linear accelerator, energy radiated per time for a given applied force
For a circular accelerator:

\[ \frac{d \vec{P}^\mu}{d \tau} \frac{d \vec{P}^\nu}{d \tau} = \frac{d \vec{P}^1}{d \tau} \frac{d \vec{P}^2}{d \tau} = \gamma^2 \frac{d \vec{P}}{d \tau^2} \]

Since \( \frac{d \vec{E}}{c}{d \tau^2} = \frac{1}{c} \cdot \frac{d \vec{P}}{d \tau} = 0 \)

\( \Rightarrow \) force 1 to velocity for circular motion

\[ P(T) = \frac{e^2}{2} \frac{2}{4 \pi} \frac{\gamma^2 (d \vec{P}^1)^2}{3 m^2 c^3 (d \tau)^2} \]

Thus we see that the energy loss to radiation for a given applied force is \( \gamma^2 \) larger for circular motion as opposed to linear motion.
Problem

\[ A^\mu = \frac{d^2 x^\mu}{d\tau^2} \] - acceleration in rest frame

In rest frame

\[ A^\mu = \begin{pmatrix} 0 \\ \alpha_\parallel \\ \alpha_\perp \end{pmatrix} \]

Find the relation between \( \alpha_\parallel \) and \( \alpha_\perp \) and \( \alpha_\parallel \) and \( \alpha_\perp \)

Solution:

Boosting

\[ \frac{du^0}{d\tau} = A^0 = \gamma \beta \alpha_\parallel \]
\[ \frac{du^\parallel}{d\tau} = A^\parallel = \gamma \alpha_\parallel \]
\[ \frac{du^\perp}{d\tau} = A^\perp = \alpha_\perp \Rightarrow \alpha_\perp = \frac{d^2 x^\perp}{d\tau^2} = \frac{d^2 \delta^\perp}{d\tau^2} = \gamma^2 \alpha_\parallel \]

\[ \alpha_\parallel \]

\[ \frac{\alpha_\parallel}{\delta^2} = \alpha_\perp \]
Similarly

\[ \gamma \frac{\partial v_{\|}}{\partial t} = \gamma \alpha_{\|} \]

Using

\[ \frac{1}{\gamma} \frac{d}{dt} \frac{v_{\|}}{\sqrt{1 - v^2/c^2}} = \frac{a_{\|}}{\gamma^2} + \frac{v_{\|}}{\gamma} \left( 1 + \frac{2v \cdot \hat{a}}{c^2} \right) \frac{1}{\sqrt{1 - (v/c)^2}} \left( 1 - \frac{v^2}{c^2} \right)^{3/2} \]

\[ = \frac{a_{\|}}{\gamma^2} \left( 1 + \frac{v^2}{c^2} \right) \]

\[ \alpha_{\|} = \frac{a_{\|}}{\gamma^3} \]
Basic Picture of Synchrotron Radiation

- Charged particle going in a circle:

  Every period the headlight goes around and shines light in your direction.

- The pulses of light you receive are very short because the collinear cone is very narrow.
Basic Use of Synchrotron Radiation

⇒ Because the pulses are very narrow in time, they contain a wide range of frequencies (useful!)

⇒ The can be very intense sources of light if the $\gamma$-factor is high

Physics Questions:

⇒ How intense?

⇒ What is the frequency distribution of the emitted light

---

Estimate of Frequency Width

⇒ What is the range of frequencies?

$$\Delta W \sim \frac{1}{\Delta t}$$

We will show that the temporal widths of the pulse is

$$\Delta = \Delta t \sim \frac{R_0}{c} \frac{1}{\gamma^3}$$
(See Figures !!!!)

1. At time $T_1$, (retarded time) the spotlight is starting to point in your angular direction. The leading pulse leaves

2. The strobe-light will point in your direction for a time set by the angular width of the beam, $\alpha$, and the angular velocity, $\omega_0$.

$$T_2 - T_1 = \frac{\alpha}{\omega_0} = \frac{R_0 \cdot \alpha}{V}$$

$$\omega_0 = \frac{R_0}{V}$$

3. At time $T_2$, the spotlight leaves your direction

4. The spatial separation between the leading edge, and the trailing edge is

$$\Delta = \frac{R_0 \cdot c}{V} - R_0 \cdot \frac{\alpha}{\beta} = R_0 \left( \frac{1}{\beta} - 1 \right) = \frac{R_0}{\beta^2}$$

distance \hspace{1cm} distance

light \hspace{1cm} particle

moved \hspace{1cm} moves
Since the angular width of the beam is \( \sim \frac{1}{\gamma} \)

the time width is

\[
\Delta t \sim \Delta \sim \frac{R_0 \alpha}{c} \frac{1}{\gamma^2}
\]

\[
\Delta t \sim \Delta \sim \frac{R_0}{c} \frac{1}{\gamma^3}
\]

So

\[
\Delta W \sim \gamma^3 \frac{c}{R_0}
\]
Figure Credit,
Christina Athanasision, et al.
arXiv:1001.3880

\[ \Delta \sim \frac{R_0 \Delta t}{\Delta} \sim \frac{R_0 (1 - \beta)}{\gamma} \sim \frac{R_0}{\gamma^3} \]

\[ \alpha = \text{angular width of cone-like radiation from each instant} \]

\[ \equiv \Delta \theta \sim \frac{1}{\gamma} \]
Figure Credit, Christina Athanasiou et al, arxiv: 1001.3880