Last Time

- Discussed radiation from charged particles

(1) $-\Box A^{\mu} = \frac{J^{\mu}}{c}$

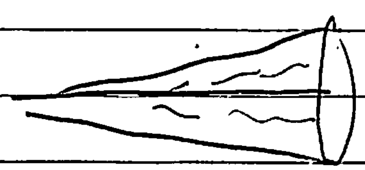
(2) $$\varphi = \frac{e}{4\pi R\,(1 - n\cdot\beta(T))}$$

$$\vec{A} = \frac{e\,\vec{V}(T)}{4\pi R\,(1 - \vec{n}\cdot\beta(T))}$$

Then in the far zone we computed the electric field

$$\vec{E} = \frac{e}{4\pi c^2}\left[\frac{1}{R}\frac{n\times(n-\beta)\times\dot{a}}{(1 - n\cdot\beta)^3}\right]_{ret}$$

Qualitative Features:



(1) Radiation localized in a cone of $1/\gamma$. Can see this since the collinear factor

$$\frac{\Delta T}{\Delta t} = \frac{1}{(1-\vec{n}\cdot\beta)} \simeq \frac{1}{\left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right)} \simeq \frac{2\gamma^3}{(1+(\gamma\theta)^2)}$$

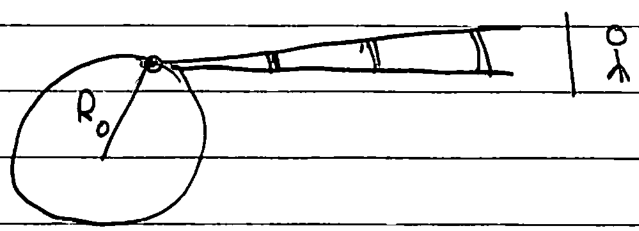
Last Time Continued

(2) We derived and discussed the Larmour formula

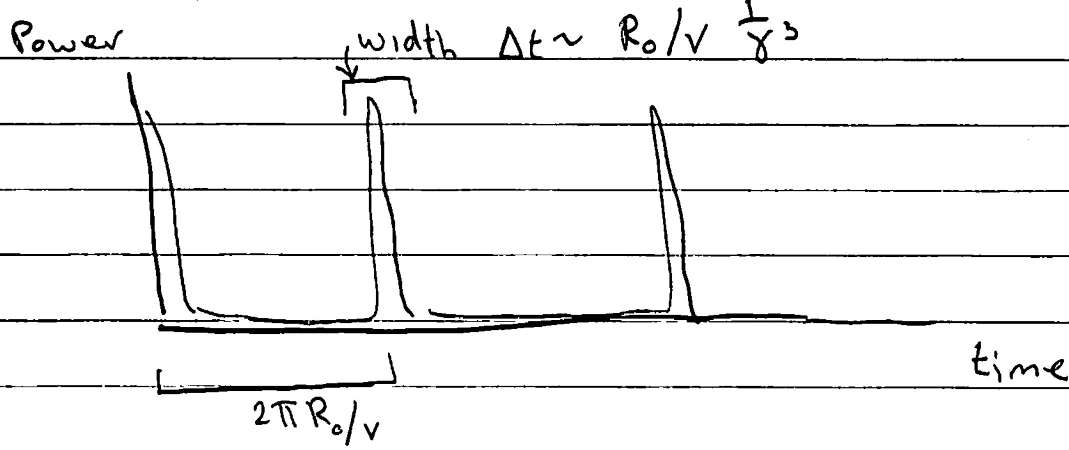
$$P = \frac{e^2}{4\pi}\frac{2}{3}\frac{1}{c^3}\gamma^6\left(a_{\parallel}^2 + \frac{a_{\perp}^2}{\gamma^2}\right)$$

$$= \frac{e^2}{4\pi}\frac{2}{3c^2}A^\mu A_\mu \qquad A^\mu = \frac{dU^\mu}{d\tau} = \frac{1}{m}\frac{dp^\mu}{d\tau}$$

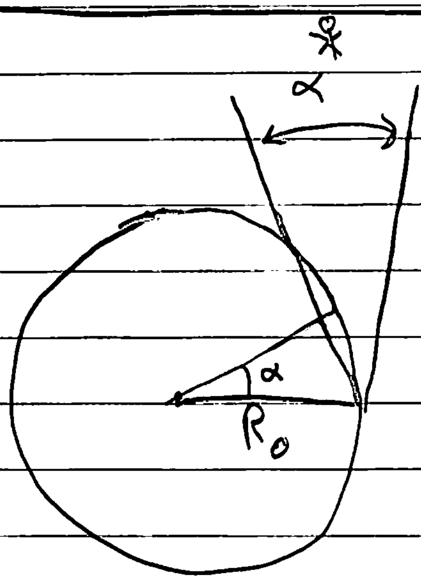
Then we started to discuss synchrotron radiation



- Each time the particle goes around a circle the strobe light points in your eye
- See blips of radiation



Review of why Δt is $\Delta t \sim \frac{R}{v\gamma^3}$



$\alpha \equiv$ Angular cone width $\sim \frac{1}{\gamma}$

ΔT = time light in your direction

$$= \frac{\omega_0}{\alpha}$$ ← angular velocity

$\alpha \sim 1/\gamma$

$$\Delta T \sim \frac{R_0}{v\gamma}$$

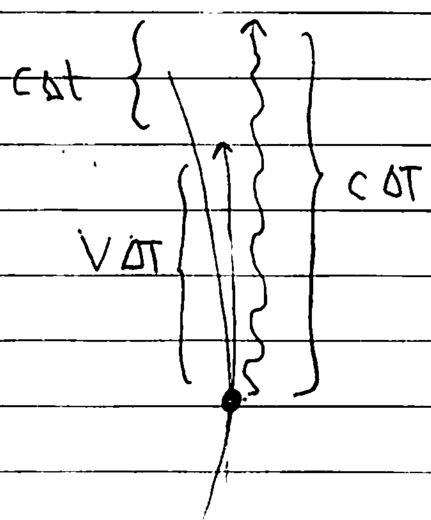
Now the emitted light fronts are separated by Δt:

$$\Delta t = \frac{\Delta t}{\Delta T}\,\Delta T$$

$$= (1 - n \cdot \beta)\,\Delta T$$

$$\simeq \left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right)\frac{R_0/v}{\gamma} \simeq \underbrace{(1+(\gamma\theta)^2)}_{\sim 1}\,\underbrace{\frac{R_0/v}{2\gamma^3}}_{\sim R_0/v\gamma^3}$$

Picture:



$$c\Delta t \simeq (c - v)\,\Delta T$$

$$\therefore \frac{\Delta t}{\Delta T} \simeq (1-\beta) \simeq \frac{1}{2\gamma^2}$$

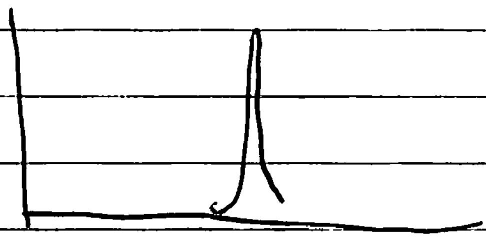
Review of Why $\omega_{typ} \sim \gamma^3 c/R$

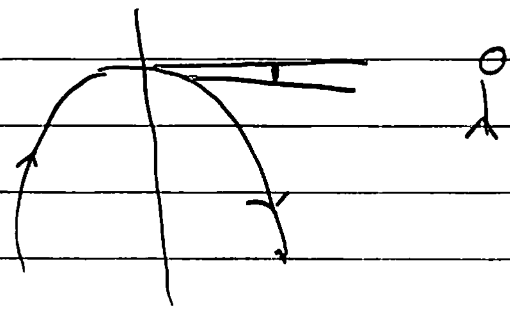
This short temporal pulse of radiation implies a broad frequency spectrum

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \frac{v\gamma^3}{R} \simeq \frac{c\gamma^3}{R}$$

Fourier Spectrum

Take a single pulse and fourier analyze :





$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \overbrace{c\,|R\,E(t)|^2}^{c R^2 \vec{S}\cdot\vec{n}} \, dt$$ ← energy per solid angle in pulse

$$\frac{dW}{d\Omega} = c \int_{-\infty}^{\infty} |R\,\vec{E}(t)|^2 \, dt$$

Using Parsevals theorem (Proved in past Home-work)

$$\int_{-\infty}^{\infty} |RE(t)|^2 \, dt = \int_{-\infty}^{\infty} |R\,\vec{E}(\omega)|^2 \frac{d\omega}{2\pi}$$

Where

$$RE(\omega) = \int_{-\infty}^{\infty} e^{+i\omega t} RE(t) \, dt$$

We have

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, c\,|R\,\vec{E}(\omega)|^2$$

We have then

$$\boxed{2\pi \frac{dW}{d\omega\, d\Omega} \equiv c\,|R\vec{E}(\omega)|^2}$$

Sometimes define

$$\frac{dI}{d\omega\, d\Omega} = \frac{c}{2\pi}\left(|R\vec{E}(\omega)|^2 + \overbrace{|R\vec{E}(-\omega)|^2}^{\text{same as } |R\vec{E}(\omega)|^2}\right)$$

since $\vec{E}^*(\omega) = \vec{E}(-\omega)$ for real $R\vec{E}(t)$

$$\frac{dI}{d\omega\, d\Omega} = \frac{c}{\pi}\,|R\vec{E}(\omega)|^2$$

Then

$$W = \int_0^\infty \frac{dI}{d\omega\, d\Omega}\, d\omega$$

The Spectrum

$$\vec{RE}(\omega) = \frac{q}{4\pi c^2}\int_{-\infty}^{\infty} e^{i\omega t} \underbrace{\left[\frac{\vec{n}\times(\vec{n}-\vec{\beta})\times\vec{a}}{(1-n\cdot\beta)^3}\right]_{ret}}_{\text{everything here is measured with } T(t,r)} dt$$

Changing variables to T:

$$T = t - \frac{|\vec{r}-\vec{r}_*(T)|}{c}$$

$$T \simeq t - \frac{r}{c} + \vec{n}\cdot\frac{\vec{r}_*}{c}(T)$$

So

$$t = T + \frac{r}{c} - \frac{n\cdot r_*}{c}(T) \qquad \text{and} \qquad \frac{dt}{dT} = (1-n\cdot\beta)$$

We have

$$\vec{RE}(\omega) = \frac{q\, e^{i\omega\frac{r}{c}}}{4\pi c^2}\int_{-\infty}^{\infty} dT\, e^{i\omega(T - n\cdot\frac{r_*}{c}(T))} \left[\frac{n\times(\vec{n}-\vec{\beta})\times\vec{a}(T)}{(1-n\cdot\beta)^2}\right]$$

The Spectrum pg. 2

Now we use

$$\frac{d}{dT}\frac{n\times n\times\vec{\beta}}{(1-n\cdot\beta)} = \frac{1}{c}\frac{n\times(\vec{n}-\vec{\beta})\times\vec{a}}{(1-n\cdot\beta)^2}$$

So

$$R\vec{E}(\omega) = \frac{q}{4\pi c}e^{i\omega r/c}\int_{-\infty}^{\infty}dT\, e^{+i\omega(T-n\cdot\frac{r_*}{c}(T))}\frac{d}{dT}\left[\frac{n\times n\times\vec{\beta}}{(1-n\cdot\beta)}\right]$$

Integrate by parts:

$$RE(\omega) = \frac{q}{4\pi c}e^{i\omega r/c}\,-i\omega\int_{-\infty}^{\infty}dT\, e^{i\omega(T-n\cdot\frac{r_*}{c}(T))}(\vec{n}\times\vec{n}\times\beta(T))$$

So

$$\boxed{2\pi\frac{dW}{d\omega\, d\Omega} = \frac{q^2}{16\pi^2}\frac{\omega^2}{c}\left|\int_{-\infty}^{\infty}dT\, e^{i\omega(T-n\cdot r_*(T))}\, n\times n\times\beta(T)\right|^2}$$

So the energy spectrum is determined by a retarded fourier transfor of the transverse current:

$$\swarrow \text{transvese current}$$

$$\frac{J_t}{c} = \vec{n}\times\vec{n}\times eV/c$$