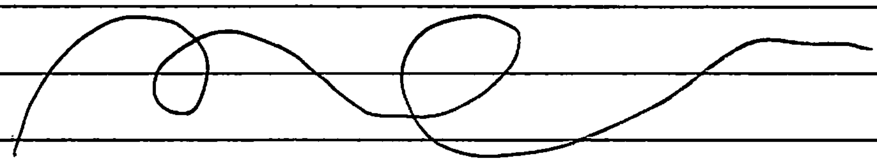
## Last Time

Determined the power spectrum for arbitrary relativistic motion:



Alternate Derivation: (Sketch)

$$\vec{A}_{rad} = \frac{1}{4\pi r}\int_{r_0} \frac{\vec{J}}{c}(T, r_0)$$ ⟸ formula from attena theory

$$T = t - \frac{r}{c} + \frac{\vec{n}\cdot r_0}{c}$$

function of $r_0$

Putting $\vec{J}(T, r_0) = q\,V(T)\,\delta^3(r_0 - r_*(T))$ and

doing the $d^3 r_0$ integral, $\delta^3(r_0 - r_*(T)) = \delta(r_0 - a)/(1 - n\cdot\beta)$

$$\vec{A}_{rad} = \frac{1}{4\pi r}\,\frac{q\,V(T)/c}{(1 - n\cdot\beta(T))}$$

$$r A_{rad} = \frac{q}{4\pi}\,V(T)/c\,\frac{dT}{dt}$$

# Last Time Continued

Then

$$E_{rad} = n \times n \times \frac{1}{c}\frac{\partial A_{rad}}{\partial t}$$

$$= n \times n \times \frac{1}{c}\frac{\partial T}{\partial t}\frac{\partial A_{rad}}{\partial T}$$

$$\downarrow$$

$$E_{rad} = \frac{q}{4\pi r c^2}\frac{1}{(1-n\cdot\beta)}\frac{n \times (\vec{n}-\vec{\beta}) \times \vec{a}}{(1-n\cdot\beta)^2}$$

Then

$$rE_{rad}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\, n \times n \times \frac{\partial A_{rad}}{c\,\partial t}\, dt$$

$$= \frac{-i\omega}{c}\int_{-\infty}^{\infty} n \times n \times A_{rad}\, e^{i\omega t}\, dt$$

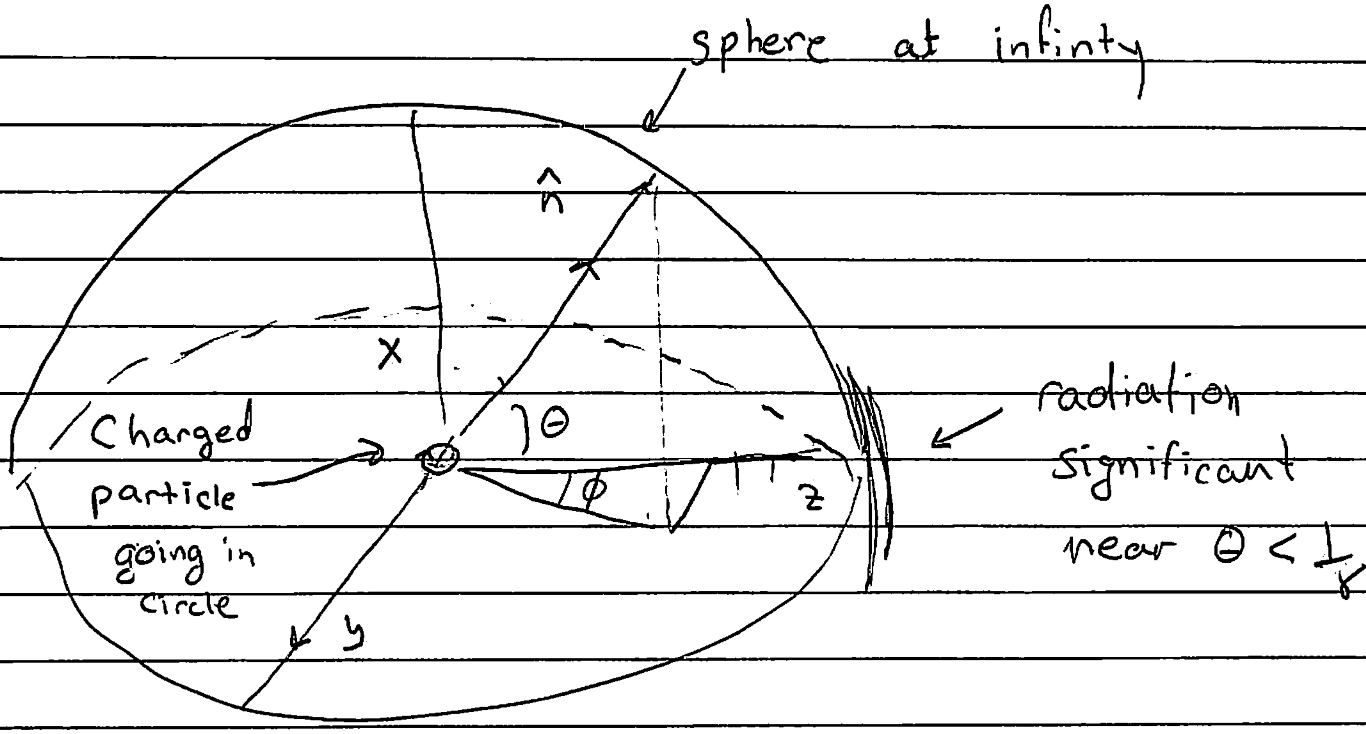
$$= -i\omega \int_{\infty}^{\infty} n \times n \times \frac{eV}{4\pi}(T)/c\, e^{i\omega t}\, \frac{dT}{dt}\, dt$$

$$\boxed{rE_{rad}(\omega) = \frac{e}{4\pi c^2}\, -i\omega \int_{-\infty}^{\infty} n \times n \times V(T)\, e^{i\omega(T + \frac{r}{c} - n\cdot r_*)}\, dT}$$

So,

$$\boxed{2\pi \frac{dN}{d\omega\, d\Omega} = c\, |rE_{rad}(\omega)|^2}$$

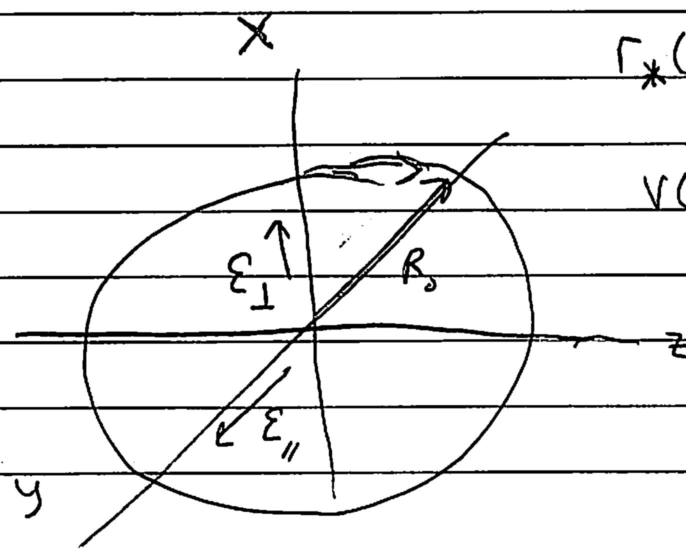
# Doing The Integral




(1) Choose without loss of generality $\hat{n}$ on at $\phi = 0$. (The spectrum is the same at any $\phi$)

$$\hat{n} = (\sin\theta, 0, \cos\theta)$$

Particles trajectory lies in the z, y plane:

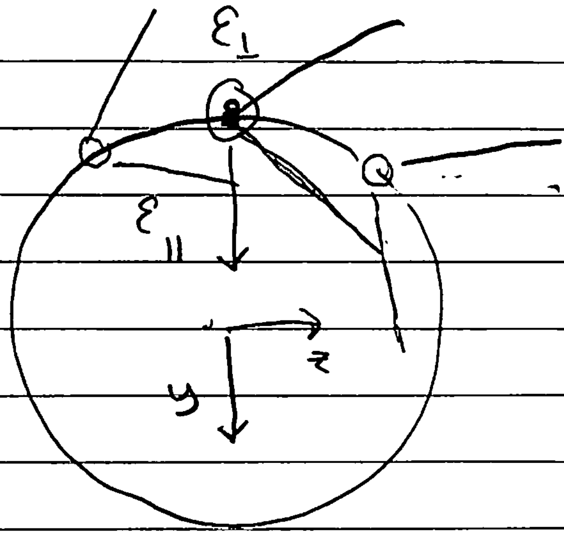


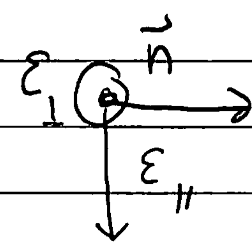

$$r_*(t) = (0, -R_0\cos\omega_0 t, R_0 \sin\omega_0 t)$$

$$V(t) = V_0 (0, \sin\omega_0 t, \cos\omega_0 t)$$

# Doing the Synchrotron Integral pg. 2

So the top view




Then we consider a time near $t \simeq 0$ and $\theta \simeq 0$.

So, $\vec{n} \times \vec{n} \times \vec{\beta} = -\vec{\beta} + \vec{n}\,(\vec{n} \cdot \vec{\beta})$

and
$$= \frac{v_0}{c}\left(\cos\omega_0 T \cos\theta \sin\theta,\ -v_0 \sin\omega_0 T,\ -v_0 \cos(\omega_0 T)\sin^2\theta\right)$$

$$\simeq \frac{v_0}{c}\left(\theta,\ -\omega_0 T,\ 0\right)$$

$$= \frac{v_0}{c}\,\theta\,\vec{\mathcal{E}}_\perp + -\frac{v_0}{c}\,\omega_0 T\,\vec{\mathcal{E}}_\parallel$$

$$n \times (n \times \vec{\beta}) \simeq \theta\,\vec{\mathcal{E}}_\perp + -\frac{cT}{R_0}\,\vec{\mathcal{E}}_\parallel$$

Similarly we approximate the phase

$$\phi = \omega\left(T - \frac{n \cdot r_*(T)}{c}\right) = \omega\left(T - \frac{R_0 \sin\omega_0 T \cos\Theta}{c}\right)$$

Then look at the phase which vanishes at $T=0$ and expanding in time $v_0 \approx \omega_0 R$

$$\phi = \omega\left(\overbrace{\left(1 - \frac{v_0}{c}\cos\Theta\right)}^{\text{cancellation}} T + O(T^3)\right)$$

$$\simeq \omega\left(\left(\frac{1}{\gamma^2} + \Theta^2\right) T + O(T^3)\right)$$

But, $\frac{1}{\gamma^2} T \sim O(T^3)$ so we should keep $T^3$ terms:

$$\phi \simeq \omega\left(\left(\frac{1}{2\gamma^2} + \frac{\Theta^2}{2}\right) T + \frac{1}{3!} \cdot \frac{c^2}{R_0^2} T^3\right)$$

$$\sin\omega_0 T \simeq \underset{\substack{\uparrow \\ \text{Straight line}}}{\omega_0 T} - \underset{\substack{\uparrow \\ \text{acceler}}}{\frac{1}{3!}(\omega_0 T)^3}$$

$$\phi \simeq \frac{\omega}{2}\left(\left(\frac{1}{\gamma^2} + \Theta^2\right) T + \frac{1}{3}\frac{c^2 T^3}{R_0^2}\right)$$

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So with our formulas

$$2\pi \frac{dW}{d\omega\, d\Omega} = c \left| r\, E(\omega) \right|^2$$

With

$$E(\omega) = \frac{q\, e^{i\omega r/c}}{4\pi r c} \left[ -i\omega \int_{-\infty}^{\infty} dT\, e^{+i\omega (T - \vec{n}\cdot r_*(T)/c)}\, \vec{n} \times (\vec{n} \times \vec{\beta}(T)) \right]$$

$$\frac{2\pi\, dW}{d\omega\, d\Omega} = \frac{q^2}{16\pi^2} \frac{\omega^2}{c} \left| \int_{-\infty}^{\infty} dT\; e^{i\omega (T - \frac{\vec{n}\cdot\vec{r}_*(T)}{c})}\; \vec{n} \times \vec{n} \times \vec{\beta}(T) \right|^2$$

Find

$$2\pi \frac{dW}{d\omega\, d\Omega} = \frac{q^2}{16\pi^2} \frac{\omega^2}{c} \left| \overbrace{A_\perp}^{\propto\, \Theta}(\omega)\, \mathcal{E}_\perp + \overbrace{-A_\parallel(\omega)}^{\propto\, cT/R_0}\, \vec{\mathcal{E}}_\parallel \right|^2$$

$$A_\perp = \Theta \int_{-\infty}^{\infty} dT\; e^{i\frac{\omega}{2}\left\{ \left(\frac{1}{\gamma^2} + \Theta^2\right) T + \frac{1}{3} \frac{c^2}{R_0^3} T^3 \right\}}$$

$$A_\parallel = \int_{-\infty}^{\infty} dT\; e^{i\omega/2 \left\{ \left(\frac{1}{\gamma^2} + \Theta^2\right) T + \frac{1}{3} c^2/R^2\, T^3 \right\}} \frac{cT}{R_0}$$

Now rescale the parameter

$$x = \underbrace{\frac{cT}{R}}_{\text{dimension}} \underbrace{\frac{1}{(1/\gamma^2 + \Theta^2)}}_{\text{appears in exponent.}}$$

define $\xi = \frac{\omega R_0/c}{3\gamma^3} \left(1 + (\gamma\Theta)^2\right)^{3/2}$

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Then

$$A_{\parallel} = \overbrace{\frac{R_0}{c}\left(\frac{1}{\gamma^2}+\theta^2\right)}^{\propto\, \xi^{2/3}} \underbrace{\int_{-\infty}^{\infty} x \exp\left(i\frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)\right) dx}_{\frac{2}{\sqrt{3}} K_{2/3}(\xi)}$$

← airy integral of Modified Bessel

$$A_{\perp} = \frac{R_0\theta}{c} \overbrace{\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}}^{\propto\, \xi^{1/3}} \underbrace{\int_{-\infty}^{\infty} \exp\left(i\frac{3}{2}\xi\left(x+\frac{1}{3}x^3\right)\right) dx}_{\frac{2}{\sqrt{3}} K_{1/3}(\xi)}$$

So then

$$\boxed{2\pi \frac{dW}{d\omega d\Omega} = \frac{3q^2\gamma^2}{4\pi^2 c}\left[\overbrace{\left(\frac{\omega R_0}{3c\gamma^3}\right)^{2/3}\left(\xi^{2/3}K_{2/3}(\xi)\right)^2}^{\text{parallel piece}} \underbrace{\left(\frac{\omega R_0}{3c\gamma^3}\right)^{4/3}\left(\gamma\theta\, \xi^{1/3} K_{1/3}(\xi)\right)^2}_{\text{perpendicular piece}}\right]}$$

↳ A wonderful but complex formula

$$2\pi \frac{dW}{d\omega d\Omega} = \frac{q^2\gamma^2}{c} F\left(\frac{\omega R_0}{c\gamma^3}, \gamma\theta\right) \qquad \xi \equiv \frac{\omega_0 R}{3c\gamma^3}\left(1+(\gamma\theta)^2\right)^{3/2}$$

Then with straightforward algegra:

$$\omega^2 A_{\parallel}^2 = \gamma^2 \left(\frac{\omega R_0}{c\gamma^3}\right)^2 (1+(\gamma\theta)^2)^2 \quad \frac{4}{3} K_{2/3}^2(\xi)$$

$$= \gamma^2 \frac{4}{3} \cdot 9 \left( \left(\frac{\omega R_0}{3\gamma^3 c}\right)^2 (1+(\gamma\theta)^2)^2 \quad K_{2/3}^2 \right)$$

$$= \gamma^2 \cdot 12 \cdot \left[ \left(\frac{\omega R_0}{3\gamma^3 c}\right)^{2/3} \left( \xi^{4/3} K_{2/3}^2 \right) \right]$$

we used

$$\xi = \frac{\omega R_0}{3\gamma^3 c} \left(1 + (\gamma\theta)^2\right)^{3/2}$$

$$\xi^{4/3} = \left(\frac{\omega R_0}{3\gamma^3 c}\right)^{4/3} \left(1 + (\gamma\theta)^2\right)^2$$

Similarly

$$\omega^2 A_{\perp}^2 = \left(\frac{\omega R_0}{c}\right)^2 \theta^2 \left(\frac{1}{\gamma^2} + \theta^2\right) \quad \frac{4}{3} \left(K_{1/3}(\xi)\right)^2$$

$$= \frac{4}{3} \cdot 9 \gamma^2 \left(\frac{\omega R_0}{3c\gamma^3}\right)^2 \left(1 + (\gamma\theta)^2\right) \quad K_{1/3}^2$$

Some - Algebra pg. 2 Skipped in class

So

$$\omega^2 A_\perp^2 = 12\,\gamma^2 \left(\frac{\omega R_o}{3c\gamma^3}\right)^{4/3} \left(\left(\left(\frac{\omega R_o}{3c\gamma^3}\right)(1+(\gamma\theta)^2)^{3/2}\right)^{2/3} K_{1/3}^2\right)$$

$$= 12\gamma^2 \left(\frac{\omega R_o}{3c\gamma^3}\right)^{4/3} \left(\xi^{1/3} K_{1/3}\right)^2$$

So

$$2\pi \frac{dW}{d\omega\, d\Omega} = \frac{q^2}{16\pi^2 c}\, \omega^2 \left[ A_\perp^2 + A_\parallel^2 \right]$$

$$= \frac{3}{4}\frac{q^2\gamma^2}{\pi^2 c}\left[\left(\frac{\omega R_o}{3c\gamma^3}\right)^{2/3}\left(\xi^{2/3} K_{2/3}\right)^2 + \left(\frac{\omega R_o}{3c\gamma^3}\right)^{4/3}(\gamma\theta)^2\left(\xi^{1/3} K_{1/3}\right)^2\right]$$

(1) Now lets analyze this formula by considering in plane emission $\theta \simeq 0$ :

Then

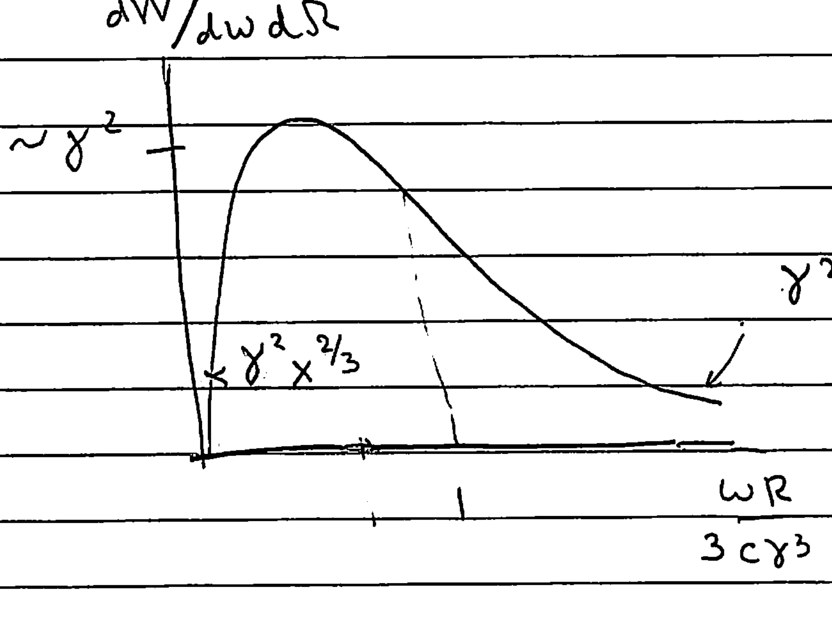
$$\left. 2\pi \frac{dW}{d\omega\, d\Omega} \right|_{\theta \simeq 0} = \frac{3 q^2 \gamma^2}{4\pi^2 c} \left[ \left( \frac{\omega R_0}{3 c \gamma^3} \right)^{2/3} \cdot \left( \xi^{2/3} K_{2/3}(\xi) \right)^2 \right]$$

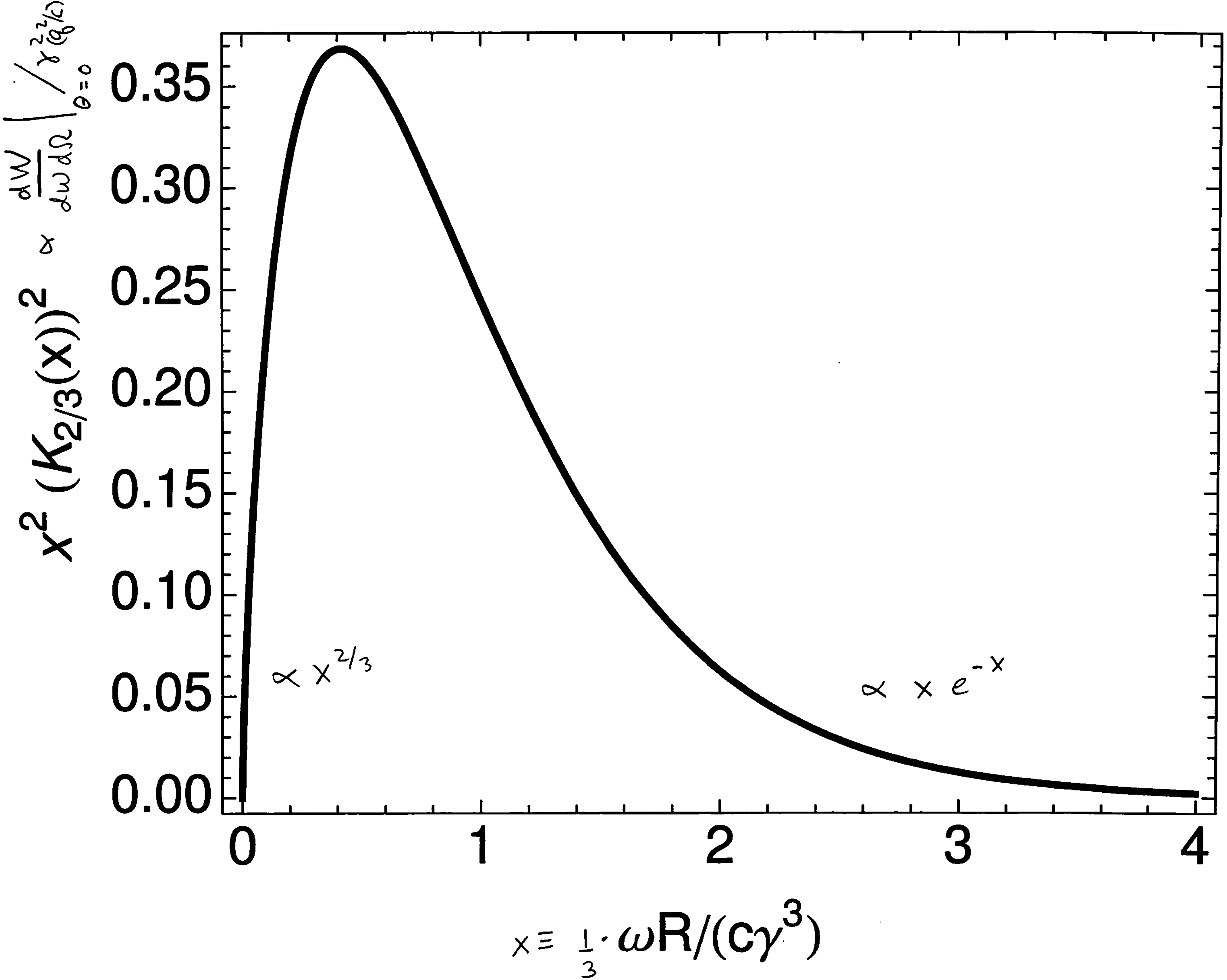
$$\xi = \frac{\omega R_0}{3c} \left( \frac{1}{\gamma^2} + \theta^2 \right)^{3/2} \xrightarrow[\theta \to 0]{} \frac{\omega R_0}{3 c \gamma^3} \equiv x$$

So for $\theta = 0$ we have:

$$2\pi \frac{dW}{d\omega\, d\Omega} = \frac{3 q^2}{4\pi^2 c} \gamma^2 \left[ x^2 \left( K_{2/3}(x) \right)^2 \right]$$

So

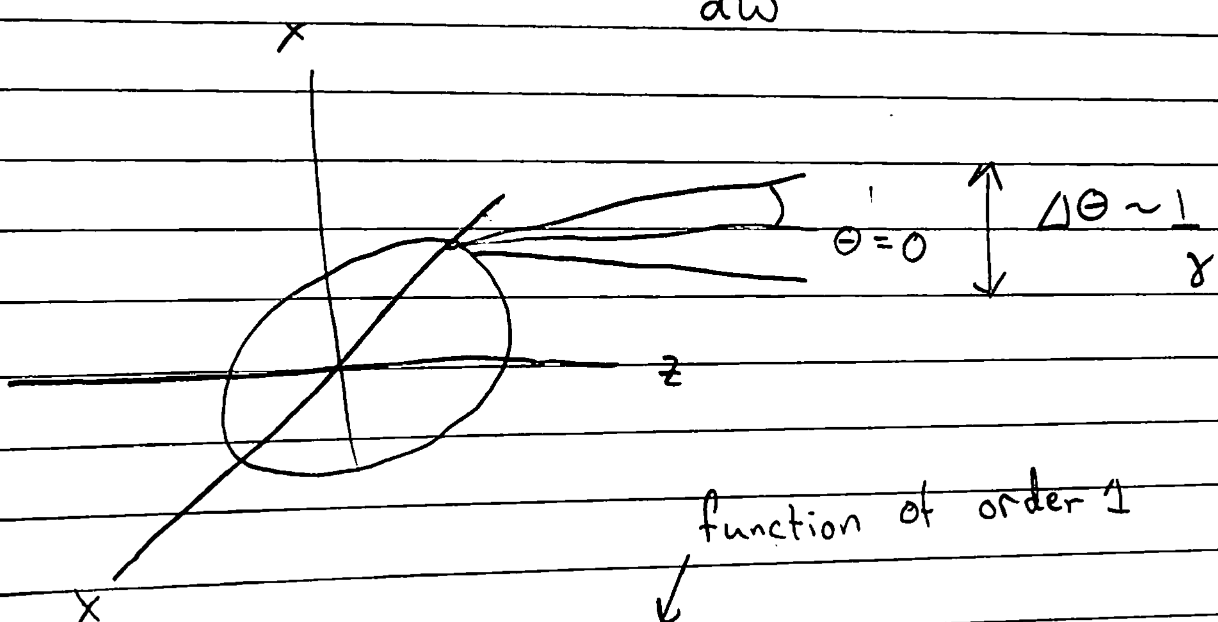



$x^2\,(K_{2/3}(x))^2 \propto \frac{dW}{d\omega\, d\Omega}\Big|_{\theta=0} \Big/ \gamma^2 (q_0^2/c)$
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
$\propto x^{2/3}$
$\propto x\, e^{-x}$
0
1
2
3
4
$x \equiv \frac{1}{3} \cdot \omega R/(c\gamma^3)$

So we see as anticipated that the typical frequency that is emitted is of order

$$\boxed{\omega \sim \frac{1}{\Delta t} \sim \frac{c\gamma^3}{R_0}} \qquad \text{or} \quad x \sim 1$$

(2) Now lets estimate $\frac{dW}{d\omega}$ and $W$: forgetting all $2\pi$'s and the like




function of order 1

$$\frac{dW}{d\omega\, d\Omega} \sim \frac{q^2\gamma^2}{c} F(x) \qquad x \equiv \frac{\omega R}{\gamma^3 c}$$

So
$$\frac{dW}{d\omega} \sim \frac{q^2\gamma^2}{c} F(x) \underbrace{\frac{1}{\gamma}}_{2\pi\Delta\theta}$$

$$\frac{dW}{d\omega} \sim \frac{q^2\gamma}{c} F(x)$$

another function of order 1

Then

$$W \sim \int \frac{dW}{d\omega}\, d\omega \qquad \text{with} \qquad x \equiv \frac{\omega R_0}{c\gamma^3}$$

$$\sim \int q^2 \frac{\gamma}{c}\, F(x)\, dx\, \frac{c\gamma^3}{R_0}$$

$$W \sim \frac{q^2}{R_0}\gamma^4 \underbrace{\int F(x)\, dx}_{\sim 1}$$

The $\gamma^4$ could have been anticipated. This is the energy per pulse. If I multiply by the number of pulses per second, $\frac{c}{2\pi R}$, I get the power radiated during circlular motion

$$P \sim \frac{q^2}{R_0}\gamma^4 \frac{1}{R_0} c$$

$$\sim \frac{q^2}{c^3}\gamma^4 \left(\frac{c^2}{R}\right)^2$$

right leading dimension

transverse acceleration, i.e. $v^2/R$

appropriate powers of $\gamma$ for transverse accel in Larmour.

Exact Answer: $P = \frac{q^2}{4\pi} \cdot \frac{2}{3}\gamma^4 \frac{c}{R_0^2}$

Other Comments

① Airy integral allways arise during Fourier transforms at high frequency at critical points

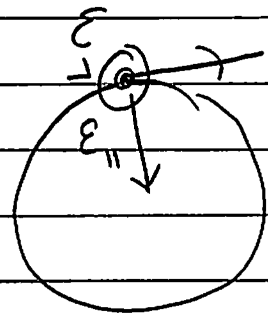
$$I = \int dt\, e^{i\omega \phi(t)} \simeq \int dt\, e^{i\omega(\alpha t + 0 + \frac{\beta}{3} t^3)}$$

Near a point of stationary phase $\phi \simeq 0$ have a taylor expansion:

$$\phi \simeq \alpha t + \underbrace{0}_{\text{no quadratic term}} + \frac{\beta}{3} t^3$$

$I$ = Airy integrals

② The polarization can be used to tell if radiation from astrophysical sources is "synchrotron like" or "thermal like"




$$\frac{P_\parallel}{P_\perp} \sim \frac{\text{power in parallel rad}}{\text{perpendic. rad}}$$

$\sim 7$

Thermal $\simeq 1$