The Polarization of the radiation

\[ \Pi \Psi = \rho \]
\[ \Pi \vec{A} = \frac{\vec{J}}{c} \]

**Setup**

**Solve**
\[ \vec{A}_{\text{rad}} = \frac{1}{4\pi} \int_{r_0}^{r} \vec{J}(T, r_0) \frac{1}{c} \]

where \( T(r) = t - \frac{r}{c} + \frac{n \cdot r}{c} \) small for non-relativistic systems

For a point-like charge
\[ \vec{J} = q \cdot V(T) \cdot \delta^3(\vec{r} - \vec{r}_0(T, r_0)) \]
we integrate over $\Gamma_0$ and find

$$A_{\text{rad}} = \frac{1}{4\pi r} \frac{q \beta (T(r_0))}{1 - n \cdot \beta (T(r_0))}$$

denominator comes from Jacobian

$$T(r_0) = \frac{t - \xi + \frac{n \cdot r_\ast (T)}{c}}{\gamma} = t_e$$

$$= \frac{\delta^3 (r_0 - r_\ast (T(r_0)))}{(1 - n \cdot \beta (T))}$$

So for non-rel systems:

$$A_{\text{rad}} = \frac{1}{4\pi r} \frac{q \beta (t_e)}{c}$$

And

$$E_{\text{rad}} = \frac{1}{4\pi rc^2} q \frac{\mathbf{n} \times \mathbf{n} \times \partial A_{\text{rad}}}{\partial t} = \frac{q}{4\pi rc^2} \mathbf{n} \times \mathbf{n} \times \partial \beta (t_e)$$

In general, decompose the outgoing radiation into its polarized components

$$\mathbf{E} = E_1 \mathbf{\xi}_1 + E_2 \mathbf{\xi}_2 \quad \mathbf{\xi}_1 \text{ and } \mathbf{\xi}_2 \text{ can be complex}$$

e.g. for circular radiation, $\mathbf{\xi} = (1, i, 0)$
Then

\[ E_1 = \hat{\epsilon}_i \cdot \vec{E} \]

\[ E_2 = \hat{\epsilon}_2 \cdot \vec{E} \]

Then the power radiated into light at a definite polarization

\[ \frac{dP(\epsilon_i)}{d\Omega} = c \left| \vec{E} \right| \left( \hat{\epsilon}_i \cdot \vec{E} \right) \]

energy per solid angle with polarization \( \epsilon_i \)

\[ \frac{dP(\epsilon_2)}{d\Omega} = c \left| \vec{E} \right|^2 \]

and

\[ \frac{dP}{d\Omega} = c \left| \vec{E} \right|^2 = c \left| \epsilon_1 \right|^2 + c \left| \epsilon_2 \right|^2 = \frac{dP(\epsilon_i) + dP(\epsilon_2)}{d\Omega} \]

Example

\[ \frac{dP}{d\Omega} = c \left| \vec{E} \right|^2 = c \left| \epsilon_1 \right|^2 + c \left| \epsilon_2 \right|^2 = \frac{dP(\epsilon_i) + dP(\epsilon_2)}{d\Omega} \]

A charged particle oscillating up and down on the \( z \)-axis

\[ \text{in plane } \vec{E}_i \rightarrow \vec{E} \]

\[ \text{out of plane } \vec{E}_i \]

Now

\[ \vec{E} = \frac{8}{4\pi r^2} \nabla \times \nabla \times \vec{A} \]
\[ E = \frac{q}{4\pi rc^2} \left[ -\dot{a} + \dot{n} (\hat{n} \cdot \dot{a}) \right] \]

where \( E \) is in the \( x-z \) plane.

In general, \( E \) is polarized in the plane of the acceleration and the observation vector \( n \).

\[ E = E_\parallel \dot{E}_\parallel + E_\perp \dot{E}_\perp \]

So,
\[ E_\parallel \cdot E = \text{something} \]

\[ E_\perp \cdot E = 0 \]
\( E_\perp \) is orthogonal to \( \dot{a} \) and \( \frac{\dot{a}}{n} \).

Then,
\[ \frac{dP}{ds} = 0 \]

while
\[ \dot{E}_\perp \cdot E = \frac{q}{4\pi rc^2} \left[ -\dot{a} \cdot \dot{E}_\parallel + \dot{n} \cdot \dot{E}_\parallel (n \cdot a) \right] \]

\[ = \frac{q}{2\pi rc^2} (-a \sin \theta) \]

and
\[ \frac{dP}{ds} = \frac{q}{16\pi^2 c^3} \frac{a^2 \sin^2 \theta}{ds} \]
Now so

\[
\frac{dP}{d\Omega} = \frac{dP_{\parallel}}{d\Omega} + \frac{dP_{\perp}}{d\Omega}
\]

\[
\frac{dP}{d\Omega} = \left( \frac{q^2 e^2 \sin^2 \theta}{16 \pi^2 c^3} \right) + 0
\]
Scattering of Radiation

\[ E_{\text{out}} \rightarrow E_{\text{inc}} \rightarrow \text{take electron or small sphere} \]

- Initially study the scattering of radiation by small objects \( \lambda \gg a \).

- Why does light scatter? The light induces currents in the object. The induced currents radiate.

So if you wanted to compute how much light was scattered, you should first compute how the incoming field accelerates the charges, and then compute how the accelerated charges radiate.

- For an extended object the acceleration in one point can create fields at another, which influences the other points, leading to a complicated standing wave pattern. Treat small (point-like) objects first.
Suggests two approximations/simplifications

1. Small objects and the external fields dipole can be considered constant, the scattering radiation field from one point of object can be neglected at another.

2. Weak scattering. The induced fields $\vec{E}_{\text{scatt}}$ are small compared to $\vec{E}_{\text{inc}}$. (Born Approximation)
What we want to compute - Cross Sections

\[ E_{\text{inc}} = E_0 e^{i k r - i \omega t} \]

\[ E(r) \rightarrow \text{Const} \, e^{i k r - i \omega t} \]

\[ \text{Const} = E_0 f(K) \]

\[ \vec{E} = \vec{E}_{\text{inc}} + \vec{E}_{\text{scatt}} \]

Scattering amplitude

The incoming energy is:

\[ S \cdot \hat{E} = \frac{1}{2} c |E_{\text{inc}}|^2 \left\langle \text{energy per area per time} \right\rangle \]

While the outgoing energy is:

\[ \frac{dP(\vec{E})}{d\Omega} = S \cdot \hat{n} = \frac{1}{2} c \left| \vec{E}_{\text{scatt}} \right|^2 \left\langle \text{energy per time per solid angle} \right\rangle \]

So then define the cross section:

\[ \frac{d\sigma (\vec{E}, \vec{E}_0)}{d\Omega} = \frac{4 \pi c}{\frac{1}{2} c |E_{\text{inc}}|^2} \left| \vec{E} \cdot f(K) \right|^2 \]

\[ \left\langle \text{energy scattered per solid angle (with polarization)} \right\rangle \]

\[ \text{per incoming flux, with polarization } \vec{E}_0 \]
Thomson - Scattering - Total Cross - Section

Let's compute the total cross-section for light electron scattering.

\[
\sigma = \text{Power Radiated} \cdot \frac{1}{\frac{1}{2}c \|E_{\text{inc}}\|^2} = \frac{1}{\frac{1}{2}c \|E_{\text{inc}}\|^2} \frac{\langle P \rangle}{\langle P \rangle}
\]

\[
\langle P \rangle = \int \sum_{\epsilon_a = \epsilon_1 + \epsilon_2} \frac{dP(\epsilon_a)}{d\Omega} d\Omega
\]

\[
= \int \frac{d\vec{P}}{d\Omega} d\Omega = \frac{q^2}{4\pi} \frac{2}{3} a^2 (\epsilon_c)
\]

So, could have defined all of this with cross sections (since \|E_{\text{inc}}\|^2 constant)

\[
\sigma = \int \sum_{\epsilon_a, \epsilon_c} d\sigma(\epsilon_a, \epsilon_c) d\Omega
\]
The force on the electron is determined by

\[ \dot{a} = q \frac{E_{\text{inc}} e^{-i\omega t}}{m} \]

Then

\[ a^2 = \frac{q^2}{m^2} \frac{1}{2} |E_{\text{inc}}|^2 \]

\[ \sigma = \frac{q^2}{4\pi} \frac{2}{3} \frac{q^2}{m^2} \frac{1}{2} |E_{\text{inc}}|^2 \]

\[ \sigma = \frac{8\pi}{3} \left( \frac{q^2}{4\pi mc^2} \right)^2 \equiv \frac{8\pi}{3} r_e^2 \]

This is known as the classical electron radius

\[ r_e = \frac{q^2}{4\pi mc^2} \]

\[ r_e = \left( \frac{q^2}{4\pi \hbar c} \right) \left( \frac{\hbar}{mc} \right) = \alpha \frac{\lambda_c}{\lambda} \]

\[ \frac{1}{137} \text{ Compton wavelength } \lambda_c \]
Compton Wavelength

\[
\frac{\hbar c}{m_e c^2} = \frac{197 \text{ eV} \cdot \text{nm}}{0.5 \text{ meV}} = 0.3 \times 10^{-12} \text{ m}
\]

\[\Gamma_e = 2.8 \times 10^{-15} \text{ m}\]

\[\Gamma_e = 2.8 \text{ fm}\]

And the Thomson-Cross Section is

\[
\frac{8\pi \Gamma_e^2}{3} = 66 \text{ fm}^2
\]

\[= 660 \text{ milli-barn}\]

\[= 0.660 \times 10^{-24} \text{ cm}^2\]

\[= 0.660 \text{ barns}\]