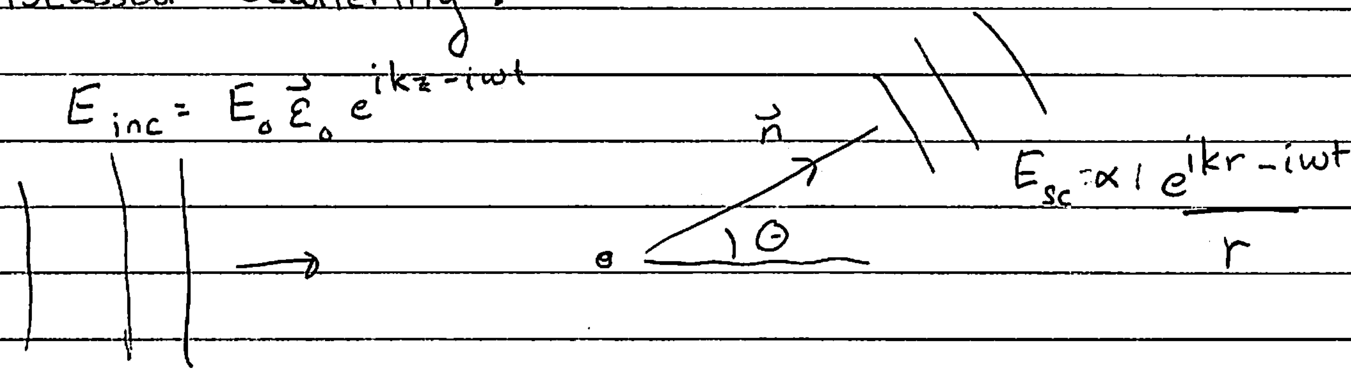
Last Time

Discussed Scattering:

$$E_{inc} = E_0 \vec{\mathcal{E}}_0 e^{ikz - i\omega t}$$



$$-\Box \vec{A} = \vec{J}/c$$

Computed $\vec{A}_{rad}$

$$A_{rad} = \frac{1}{4\pi r} \int_{r_0} J(T, r_0)/c$$

$$T = \underbrace{t - \frac{r}{c}}_{t_e} + \underbrace{\frac{\vec{n} \cdot r_0}{c}}_{\text{small}}$$

- Incoming field creates currents (Understand This first!)
- These currents radiate
  - → we will limit this to non-relativistic radiation
    - non-rel larmour
    - electric dipole, magnetic dipole

Recall

$$\vec{E}_{rad} = \frac{q}{4\pi r c^2} \vec{n} \times \vec{n} \times \vec{a}(t_e) \leftarrow \text{Larmour}$$

$$\begin{cases} \vec{E}_{rad} = \frac{1}{4\pi r c^2} \vec{n} \times \vec{n} \times \ddot{\vec{p}}(t_e) \leftarrow \text{Electric Dipole Radiation} \\ \vec{B}_{rad} = n \times E_{rad} \end{cases}$$

$$\begin{cases} B_{rad} = \frac{1}{4\pi r c^2} \vec{n} \times \vec{n} \times \ddot{\vec{m}}(t_e) \leftarrow \text{Magnetic Dipole Radiation} \\ -E_{rad} = \vec{n} \times B_{rad} \end{cases}$$

$$= -n \times \ddot{m}(te) / 4\pi r c^2$$

Can Remember magnetic dipole rad, by noting the duality transformation, $E \to B$, $\vec{B} \to -\vec{E}$

Often interested in the projection into a given polarization

$$\vec{E}_{rad} = E_1 \vec{\varepsilon}_1 + E_2 \vec{\varepsilon}_2$$

So

$$E_1 = \vec{\varepsilon}_1^{\,*} \cdot \vec{E}_{rad}$$

Note that often get stuff like

$$\vec{\varepsilon}^{\,*} \cdot (\vec{n} \times \vec{n} \times \vec{a}) = \vec{\varepsilon}^{\,*} \cdot (-\vec{a}_T)$$

This is transverse

$$= \vec{\varepsilon}^{\,*} \cdot (-\vec{a} + \vec{n}(n \cdot a))$$

$$= -\vec{\varepsilon}^{\,*} \cdot \vec{a}$$ (since ε^* is transverse to $\vec{n}$ it projects out the longitudinal part of $\vec{a}$)

Then we compute the power radiated:

$$\overline{\frac{dP(\varepsilon)}{d\Omega}} = \frac{c}{2} |r E_1|^2$$

← time ave

$$\overline{\frac{dP}{d\Omega}} = \frac{q^2}{16\pi^2 c^3} \frac{|\vec{\varepsilon}^{\,*} \cdot \vec{a}|^2}{2}$$

← time ave

For the incoming field polarized $\vec{a} = \frac{E_0}{m} e^{i\omega t} \vec{\varepsilon}_0$

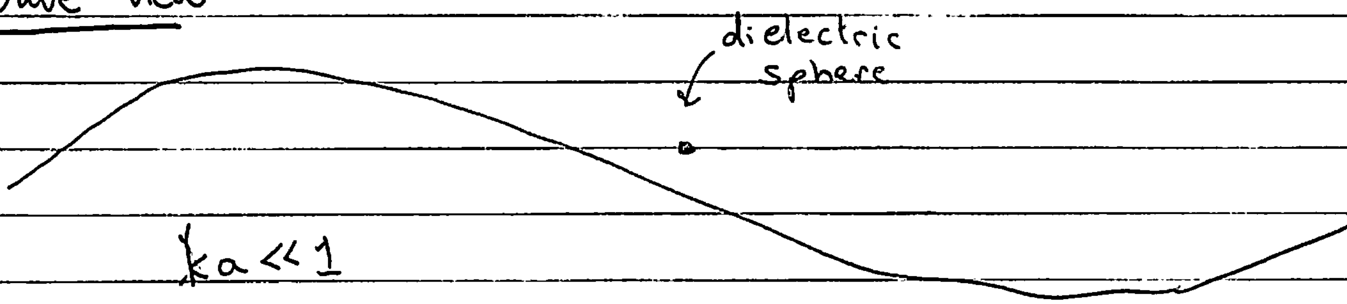
$$\frac{d\sigma(\varepsilon)}{d\Omega} = \frac{\overline{dP/d\Omega}}{\frac{c}{2}|E_0|^2} = r_e^2 |\varepsilon^* \cdot \varepsilon_0|^2$$

$$r_e \equiv \frac{e^2}{4\pi mc^2}$$

Scattering From Small Objects

- Or why the sky is blue
- what means small?

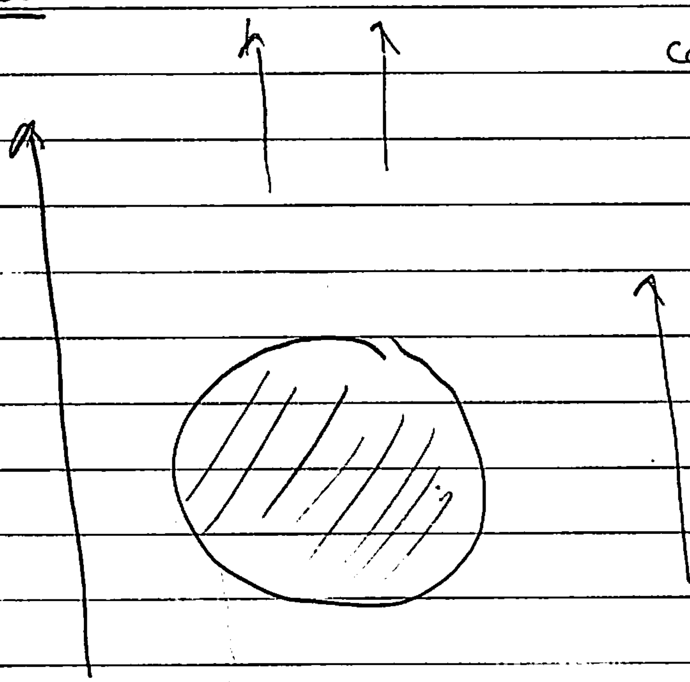
Wave view



- The small "atom" experiences an uniform electric and magnetic field. Consider a dielectric sphere

Sphere view

- Electric field constant and slowly time dependent



So

$$\frac{d\bar{P}}{d\Omega} = \frac{1}{16\pi^2} \alpha_E^2 \left(\frac{\omega}{c}\right)^4 (1 - |\varepsilon_0 \cdot n|^2) \frac{c E_0^2}{2}$$

And

$$\frac{d\sigma}{d\Omega} = \frac{\overline{dP/d\Omega}}{\frac{c}{2}|E_0|^2} = \frac{1}{16\pi^2} \alpha_E^2 \left(\frac{\omega}{c}\right)^4 (1 - |\varepsilon_0 \cdot n|^2)$$

Or using $\alpha_E = 4\pi \left(\frac{\varepsilon - 1}{\varepsilon + 2}\right) a^3$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\varepsilon - 1}{\varepsilon + 2}\right)^2 \left(\frac{\omega a}{c}\right)^4 a^2 \cdot (1 - |\varepsilon_0 \cdot n|^2)}$$

Important remark

- See a characteristic frequency dependence of dipole scattering

$$\sigma \propto \omega^4$$

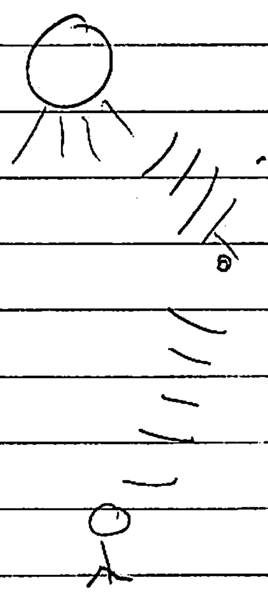
- The overall features of the cross section is the

dimension

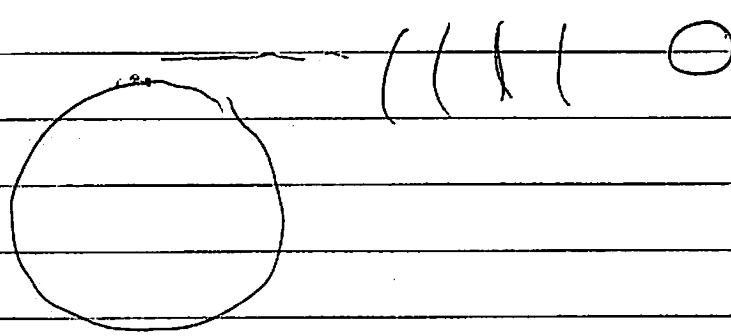
$$\sigma \propto \left(\frac{\omega a}{c}\right)^4 a^2$$

Blue-Sky and Red-sunsets

- Since the scattering cross-section is strongly peaked toward the high frequencies scattered light will appear blueish



At sunset we see the transmitted light (i.e. not scattered)



Not quite as simple as all that but it gives a qualitative picture

Polarization in Scattering

Polarization of Skylight

$$E_{rad} = \frac{1}{4\pi r c^2}\left[-\ddot{\vec{p}} + \vec{n}(n\cdot\ddot{\vec{p}})\right]$$

$$\varepsilon^* \cdot E_{rad} = \frac{1}{4\pi r c^2}\left[-\vec{\varepsilon}\cdot\ddot{\vec{p}}(t_e)\right] \qquad \vec{p} = \alpha_E \vec{E}_{inc}$$

$$t_e \equiv t - r/c$$

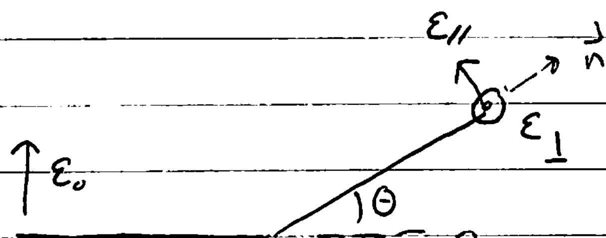
$$\varepsilon^* \cdot E_{rad} = \frac{1}{4\pi r c^2}(-\alpha_E E_0)e^{-i\omega t_e}\, \varepsilon^* \cdot \varepsilon_0 \, (-\omega^2)$$

$$\alpha_E = 4\pi\left(\frac{\varepsilon - 1}{\varepsilon + 1}\right)a^3$$

So

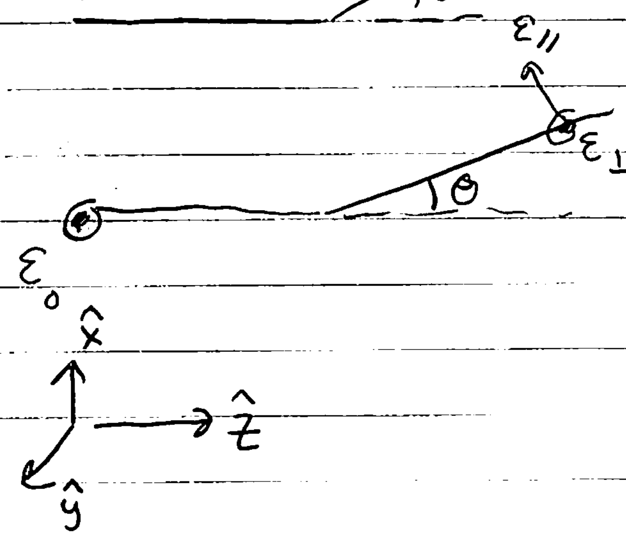
$$\frac{d\sigma}{d\Omega}(\varepsilon;\varepsilon_0) = \left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 \left(\frac{\omega a}{c}\right)^4 a^2 \left|\varepsilon^* \cdot \varepsilon_0\right|^2$$

So four cases:



Two cases

$$\left\{\begin{array}{l} \varepsilon_0 = \text{up} = \hat{x} \quad \text{(in-plane)} \\ \varepsilon = \underbrace{\varepsilon_{\parallel}}_{\text{in}} \text{ or } \underbrace{\varepsilon_{\perp}}_{\text{out}} \end{array}\right.$$



Two cases:

$$\varepsilon_0 = \text{out} = \hat{y} \quad \text{(out-plane)}$$

$$\varepsilon = \underbrace{\varepsilon_{\parallel}}_{\text{in}} \text{ or } \underbrace{\varepsilon_{\perp}}_{\text{out}}$$

Then we work out the four cases:

$$\frac{d\sigma}{d\Omega}(\varepsilon ; \varepsilon_0) \qquad C\omega^4 \equiv \left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 \left(\frac{\omega a}{c}\right)^4 a^2$$

① $\frac{d\sigma}{d\Omega}(\text{in} ; \text{in}) = C\omega^4 \, |\varepsilon_{\parallel} \cdot \hat{x}|^2 = C\omega^4 \cos^2\theta$

② $\frac{d\sigma}{d\Omega}(\text{in} ; \text{out}) =$ outgoing light polarized in plane $\varepsilon_{\parallel}$, while incoming light, ε_0, is out of plane $= 0$

③ $\frac{d\sigma}{d\Omega}(\text{out}, \text{in}) = 0$

④ $\frac{d\sigma}{d\Omega}(\text{out}, \text{out}) = C\omega^4$

- So the cross section for unpolarized incoming light to produce outgoing light in-plane, $\varepsilon_{\parallel}$, is the average of ① and ②

$$d\sigma_{\parallel}/d\Omega = \frac{① + ②}{2} = \frac{1}{2} C\omega^4 \cos^2\theta$$

- The cross section for unpolarized incoming light to produce outgoing light polarized out-of-plane, $\varepsilon_{\perp}$, is

$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{③ + ④}{2} = \frac{1}{2} C\omega^4$$

And finally the cross section for unpolarized light to produce light of any polarization is

$$\frac{d\sigma}{d\Omega} = C\omega^4 \left(\frac{1+\cos^2\theta}{2}\right)$$