Last Time:

- Green fcns in 1D

$$\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] g(x,x_0) = \delta(x-x_0)$$

Said

$$g(x,x_0) = C\left[\, y_o(x)\, y_{in}(x_0)\,\Theta(x-x_0) + y_{in}(x)\, y_o(x_0)\,\Theta(x_0-x)\right]$$

$$\equiv C\; y_o(x_>)\, y_{in}(x_<)$$

C is a constant $\Rightarrow$ $C = \dfrac{1}{p(x_0)\, W(x_0)}$

where $W(x) = y_{out}\, y_{in}' - y_{in}\, y_{out}'$

- Now for Green fcns since:

$$\delta^3(\vec{r}-\vec{r}_0) = \frac{1}{r^2}\,\delta(r-r_0)\,\delta(\cos\theta-\cos\theta_0)\,\delta(\phi-\phi_0)$$

$$= \frac{1}{r^2}\,\delta(r-r_0)\sum_{\ell m} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta_0\phi_0)$$

Last Time (continued)

It make sense to expand

$$G(\vec{r},\vec{r}_0) = \sum_{\ell m} g_{\ell m}(r,r_0)\, Y_{\ell m}(\theta\phi)\, Y_{\ell m}(\theta_0\phi_0)$$

And solve for $g_{\ell m}(r,r_0)$:

$$\left[-\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \ell(\ell+1)\right] g_{\ell m}(r,r_0) = \delta(r-r_0)$$

Using the 1D technique

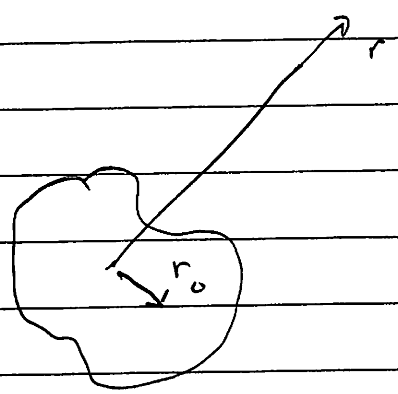
- Did this for the free space green fcn:

$$G_0(\vec{r},\vec{r}_0) = \sum_{\ell m} \frac{1}{(2\ell+1)} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta_0\phi_0)$$

We will use this stuff today!

Multipole Expansion Again:

$$\varphi(\vec{r}) = \int_{r_0} \rho(\vec{r}_0) \frac{1}{4\pi|\vec{r}-\vec{r}_0|}$$



$r_> = r$

$r_< = r_0$

Take the result

$$\frac{1}{4\pi|\vec{r}-\vec{r}_0|} = \frac{1}{2l+1} \sum_{lm} r_0^l \frac{1}{r^{l+1}} Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta_0,\phi_0)$$

Then

$$\boxed{\varphi(r) = \sum_{lm} \frac{q_{lm}}{2l+1} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}}$$

← spherical tensor form of multipole expansion

Where:

$$\boxed{q_{lm} = \int d^3\vec{r}_0\, \rho(r_0)\, Y_{lm}^*(\theta_0,\phi_0)\, r_0^l}$$

exterior spherical multipole moments

Multipoles Again (pg. 2)

So the multipoles are defined by

$$q_{\ell m} \equiv \int d^3\vec{r}_0 \; \rho(r_0)\, Y^*_{\ell m}(\theta_0, \phi_0)\, r^\ell$$

The Book defines $A_{\ell m} \equiv \frac{4\pi}{2\ell+1} q_{\ell m}$ so that the multipole expansion reads $\varphi = \sum_{\ell m} A_{\ell m} \frac{Y_{\ell m}}{4\pi r^{\ell+1}}$

Multipoles Again (pg. 3)

Then compare with the Cartesian expansions

$$\varphi(r) = \frac{Q_V}{4\pi r} + \frac{\vec{p}\cdot\hat{r}}{4\pi r^2} + \frac{\Theta_{ij}\,\hat{r}^i\hat{r}^j}{4\pi r^3} + \cdots$$

These two expansions are entirely equivalent. Comparing

$$Q_V = \frac{4\pi}{2\cdot 0+1}\, q_{00}\, Y_{00} \qquad (l=0) \qquad \text{monopole}$$

$$\vec{p}\cdot\hat{r} = \frac{4\pi}{2\cdot 1+1}\sum_{m=-1}^{1} q_{1m}\, Y_{1m} \qquad (l=1) \qquad \text{dipole}$$

$$\Theta_{ij}\,\hat{r}^i\hat{r}^j = \frac{4\pi}{2\cdot 2+1}\sum_{m=-2}^{2} q_{2m}\, Y_{2m} \qquad (l=2) \qquad \text{quadrupole mopment}$$

There is a one-to-one map between the cartesian and spherical forms:

$$Q_V \longleftrightarrow q_{00}$$

$$p_x, p_y, p_z \longleftrightarrow q_{11}, q_{10}, q_{1-1}$$

$$\underbrace{\Theta_{ij}}_{\substack{\text{symmetric traceless} \\ \text{3 by 3 matrix = 5 independent} \\ \text{component}}} \longleftrightarrow q_{22}, q_{21}, q_{20}, q_{2-1}, q_{22}$$

What's the exact relation between the two?

Lets work it out for the dipole case

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\,\frac{(x+iy)}{r}$$ see wikipedia for list up to $l=4$

$$rY_{11}^{*} = -\sqrt{\frac{3}{8\pi}}\,(x-iy)$$

Thus

$$q_{11} = \int d^3r\, \rho(r)\, \left[r\,Y_{11}^{*}\right]$$

Compare

$$= \int d^3r\, \rho(r)\, \left[-\sqrt{\frac{3}{8\pi}}\,(x-iy)\right]$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\,(p_x - ip_y)$$

In general see handout

From

Jackson, Chapter 4:

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\mathbf{x}')\, d^3x' = \frac{1}{\sqrt{4\pi}} q \tag{4.4}$$

$$\left.\begin{aligned} q_{11} &= -\sqrt{\frac{3}{8\pi}} \int (x' - iy')\rho(\mathbf{x}')\, d^3x' = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y) \\ q_{10} &= \sqrt{\frac{3}{4\pi}} \int z'\rho(\mathbf{x}')\, d^3x' = \sqrt{\frac{3}{4\pi}} p_z \end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned} q_{22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int (x' - iy')^2\rho(\mathbf{x}')\, d^3x' = \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11} - 2iQ_{12} - Q_{22}) \\ q_{21} &= -\sqrt{\frac{15}{8\pi}} \int z'(x' - iy')\rho(\mathbf{x}')\, d^3x' = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23}) \\ q_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int (3z'^2 - r'^2)\rho(\mathbf{x}')\, d^3x' = \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33} \end{aligned}\right\} \tag{4.6}$$

Only the moments with $m \geq 0$ have been given, since (3.54) shows that for a real charge density the moments with $m < 0$ are related through

$$q_{l,-m} = (-1)^m q^*_{lm} \tag{4.7}$$

Relation between cartesian and spherical multipoles:

Note: what Jackson calls Q_{ij}

$Q_{ij} \equiv 2\Theta_{ij}$ = symmetric traceless

is what we call $2\Theta_{ij}$

$$\Theta_{ij} = \frac{1}{2}\int_{\vec{r}} \rho(\vec{r})\,[3r^i r^j - r^2 \delta^{ij}] = Q_{ij}^{\text{Jackson}}/2$$