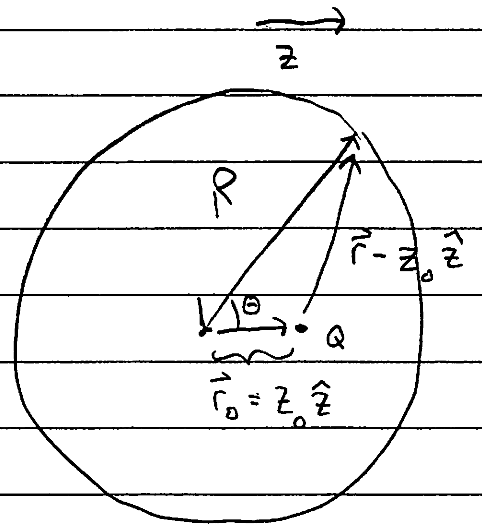
A warmup problem - pg. 1



- A charge Q in a grounded sphere of radius R, is at position $\vec{r}_0 = z_0\hat{z}$
- Determine the force on the charge

Solution:

- first set $R=1$, then note that the distance from the point charge to the sphere is

$$d(z_0,\theta) = \left.\sqrt{r^2 + z_0^2 - 2 r z_0 \cos\theta}\,\right|_{r=1} \quad \text{on surface of sphere}$$

$$= (1 + z_0^2 - 2 z_0 \cos\theta)^{1/2}$$

$$= z_0 \left(\frac{1}{z_0^2} + 1 - \frac{2}{z_0}\cos\theta\right)^{1/2}$$

$$d(z_0,\theta) = z_0\, d\left(\frac{1}{z_0}, \theta\right)$$

Thus the distance at z_0 is related to the distance at $1/z_0$

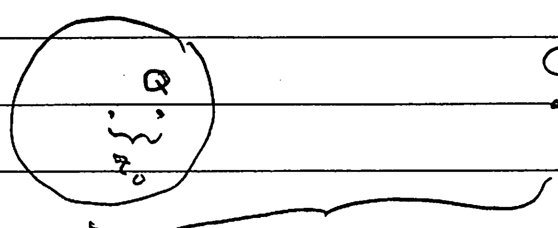
Thus:

$$\varphi(\vec{r},\vec{r}_0) = \frac{Q}{4\pi|\vec{r} - z_0\hat{z}|} - \frac{Q}{4\pi z_0 |\vec{r} - \frac{1}{z_0}\hat{z}|}$$

vanishes when $|\vec{r}| = 1$ (on the sphere)
and satisfies

$$-\nabla^2 \varphi(r, r_0) = \delta^3(\vec{r} - \vec{r}_0)$$

Picture: place an image charge $Q_I = R/z_0$ at position R^2/z_0



position $\frac{R^2}{z_0}\hat{z}$

So

$$F = Q\, E_{image}\Big|_{r = z_0\hat{z}}$$

$$F = Q \frac{Q}{4\pi|z_0\hat{z} - \frac{1}{z_0}\hat{z}|} = \frac{Q^2 z_0}{4\pi(z_0^2 - 1)}$$

For $z_0 \ll 1$ (close to center)

$$F \simeq \frac{Q^2 z_0}{4\pi}$$

Restoring units we have

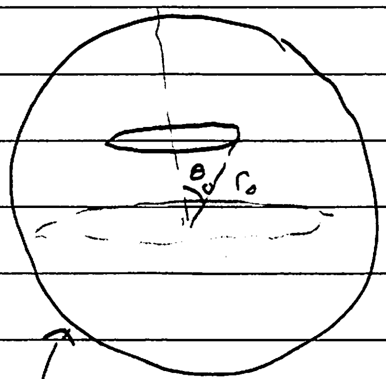
$$\boxed{F = \frac{Q^2 z_0}{4\pi R^3}}$$

For reference the Grn fcn is

$$\boxed{G(\vec{r},\vec{r}_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|} - \frac{Q\, R/r_0}{4\pi\left|\vec{r} - \frac{R^2}{r_0}\hat{r}_0\right|}}$$

A worked example pg. 1

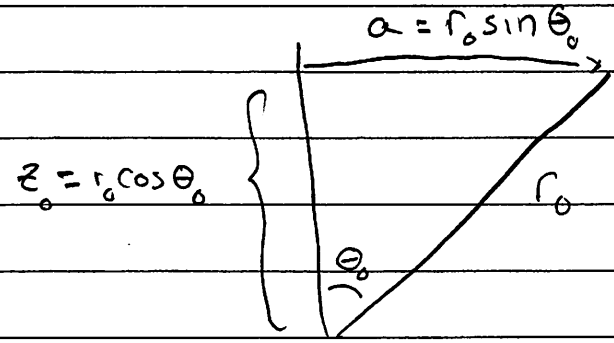
A ring of charge is held in a grounded conducting sphere.



$\varphi = 0$

on surface

- The charge per length is λ.
- The radius of the ring / sphere is a = ring and R = sphere



Determine the potential and the force on the ring.

$$\rho(\vec{r}) = \underbrace{\frac{\lambda \sin\theta_0}{r_0}\, \delta(r - r_0)\, \delta(\cos\theta - \cos\theta_0)}_{\text{a ring of charge}}$$

$$\int r^2 dr\, d(\cos\theta)\, d\phi\, \rho(r) = \int d\phi\, \lambda\, r_0 \sin\theta_0 = \lambda\, 2\pi a$$

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Now we try substituting

$$\varphi = \sum_{\ell} \hat{g}_{\ell}(r, r_0) \, P_{\ell}(\cos\theta)$$

into

$$\left[-\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} L^2 \right] \varphi = \rho(\vec{r})$$

and find

$$\sum_{\ell} P_{\ell}(\cos\theta) \left[-\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2} \right] \hat{g}_{\ell}(r, r_0) = \rho(\vec{r})$$

Now multiplying both sides by $P_{\ell'}(\cos\theta)$ and integrating with

$$\left(\frac{2\ell+1}{2} \right) \int_{-1}^{1} P_{\ell}(\cos\theta) \, P_{\ell'}(\cos\theta) = \delta_{\ell\ell'}$$

We find

$$\star \quad \left[-\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2} \right] \hat{g}_{\ell}(r, r_0) = \overbrace{\lambda a \left(\frac{2\ell+1}{2} \right) P_{\ell}(\cos\theta_0)}^{\text{~~stupid factor~~}} \delta(r - r_0$$

Thus up to a factor:

$$\hat{g}_{\ell}(r, r_0) \equiv \overbrace{\lambda a \left(\frac{2\ell+1}{2} \right) P_{\ell}(\cos\theta_0)}^{\text{~~stupid factor~~}} g_{\ell}(r, r_0)$$

$\hat{g}_{\ell}(r, r_0)$ is the green fcn. Specifically ....

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Now $g_\ell(r,r_0)$ satisfies Eq ☆ without the stupid factor

$$g_\ell(r,r_0) = C\left[y_{in}(r_0)\, y_0(r)\, \Theta(r-r_0) + y_0(r_0)\, y_i(r)\, \Theta(r_0-r)\right]$$

Where

$$C = \frac{1}{p(r_0)\, W(r_0)}$$

So

$$y_i = \left(\frac{r}{R}\right)^\ell \qquad y_{out} = A r^\ell + \frac{B}{r^{\ell+1}} = \underbrace{\left(\frac{R}{r}\right)^{\ell+1} - \left(\frac{r}{R}\right)^\ell}_{\text{vanishes at bndry}}$$

So

$$p(r_0)\, W(r_0) = r_0^2\left[y_0\, y_i' - y_i'\, y_0\right]$$

$$= (2\ell+1)R$$

So

$$g_\ell(r,r_0) = \frac{1}{(2\ell+1)R}\left(\frac{r_<}{R}\right)^\ell\left(\left(\frac{R}{r_>}\right)^{\ell+1} - \left(\frac{r_>}{R}\right)^\ell\right)$$

And

$$\boxed{\varphi = \sum_\ell \frac{\lambda q}{2R} P_\ell(\cos\theta_0)\left(\frac{r_<}{R}\right)^\ell\left(\left(\frac{R}{r_>}\right)^{\ell+1} - \left(\frac{r_>}{R}\right)^\ell\right) P_\ell(\cos}$$

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do it! $r = r_>$ $r_0 = r_<$

$$-\vec{n}\cdot\vec{\nabla}\cdot\varphi = E_n = +\left.\frac{\partial\varphi}{\partial r}\right|_{r=R}$$

$$\sigma_{ind} = -\frac{Q}{4\pi R^2}\sum_\ell P_\ell(x_0)\,P_\ell(x)\,(2\ell+1)\left(\frac{r_0}{R}\right)^\ell$$

induced charge

Now indeed:

$$Q_{ind} = \int R^2 d\Omega\,\sigma = 2\pi R^2\int_{-1}^{1} dx\left[\frac{-Q}{4\pi R^2}\sum_\ell P_\ell(x_0)\,P_\ell(x)\,(2\ell+1)\left(\frac{r_0}{R}\right)^\ell\right]$$

$$Q_{ind} = -Q \checkmark$$

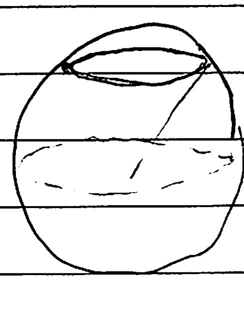
(Thus the induced charge on the sphere is $-Q$)

And for $r_0 \to R$ we have:

$$\sigma = -\frac{Q}{4\pi R^2}\sum_\ell P_\ell(x_0)\,P_\ell(x)\,(2\ell+1)$$

$$= -\frac{Q}{4\pi R^2}\,2\,\delta(x-x_0) = -\frac{Q}{2\pi R^2}\,\delta(x-x_0)$$

Thus, as the ring approaches the bndry:



then the induced charge is just a ring right where the real ring is.

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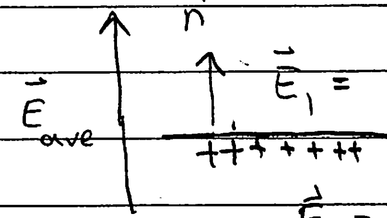
Now what is the force?

For any surface:



$\vec{E}_1 = \vec{E}_{ave} + \frac{\sigma}{2}\hat{n}$

$\vec{E}_2 = \vec{E}_{ave} - \frac{\sigma}{2}\hat{n}$

$$\vec{\mathcal{F}} = \text{force per area}$$

$$= \sigma \vec{E}_{ave}$$

$$\vec{\mathcal{F}} = \sigma \frac{1}{2}(\vec{E}_1 + \vec{E}_2)$$

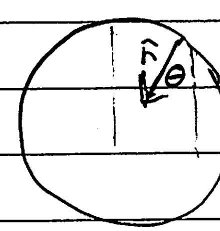
For a metal $E_2 = 0$ and $\vec{E}_1 = E_n \hat{n}$

and $\sigma = \hat{n} \cdot \vec{E}_1 = E_n$

$$\vec{\mathcal{F}} = \frac{1}{2} E_n^2 \hat{n} = \text{force per area}$$

inward directed normal

For the present case:



$$F^z_{sphere} = \int dS \, \frac{1}{2} E_n^2 \, \hat{n}^z$$

$$F^z_{sphere} = \int dS \, \frac{1}{2} E_n^2 (-\cos\theta)$$

Can also calculate the force on the ring

$$F^z_{ring} = -\int dS\, n_i^{out}\, T^{iz} = \int dS\, n_i\, T^{iz}$$

~~this~~ is outward normal

this is inward

So

$$T^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$$

$$T^{ij} = -E_n E_n \hat{n}^i \hat{n}^j + \frac{1}{2} E_n^2 \delta^{ij}$$

So

$$n_i T^{iz} = -E_n^2 \hat{n}^z + \frac{1}{2} E_n^2 \hat{n}^z$$

$$= -\frac{1}{2} E_n^2 \hat{n}^z = +\frac{1}{2} E_n^2 \cos\theta$$

So

$$F^z_{ring} = -F^z_{sphere} = \int dS\, \frac{1}{2} E_n^2 \cos\theta$$

Now for the current case

$$E_n = \frac{Q}{4\pi R^2} \sum_l P_l(x_0)\, P_l(x)\, (2l+1) \left(\frac{r_0}{R}\right)^l$$

$$F^z = \int dS \, \frac{1}{2} E_n^2 \cos\theta$$

$$F^z = \frac{Q^2}{(4\pi R^2)^2} \cdot 2\pi R^2 \int_{-1}^{1} dx \; \frac{1}{2} \sum_l \sum_{l'} P_l(x_0) P_{l'}(x_0)\, (2l+1)(2l+1)' \left(\frac{r_0}{R}\right)^l \left(\frac{r_0}{R}\right)^{l'} \left[P_l(x) P_{l'}(x)\, x \right]$$

Using:

$$(2l'+1) * P_{l'}(x) = (l'+1)\, P_{l'+1}(x) + l'\, P_{l'-1}(x)$$

And writing $\frac{1}{2}\sum_l \sum_{l'} = \sum_l \sum_{l'<l}$ we see that the sum sets $l' = l-1$ yielding:

$$F^z = \frac{Q^2}{4\pi R^2} \cdot \int_{-1}^{1} dx \sum_l P_l(x_0) P_{l-1}(x_0) \left(\frac{2l+1}{2}\right) \left(\frac{r_0}{R}\right)^{2l-1} \left[P_l(x) P_l(x)\, l \right]$$

$$\boxed{F^z = \frac{Q^2}{4\pi R^2} \sum_l P_l(x_0)\, P_{l-1}(x_0)\, l \left(\frac{r_0}{R}\right)^{2l-1}}$$

A worked example

For small sizes the lowest

$l = 1$ dominates the sum

And $P_1(x_0) = x_0$ $P_0(x_0) = 1$

$$F^z \simeq \frac{Q^2}{4\pi R^2} \dot{x}_0 \left(\frac{r_0}{R}\right)$$

$$\boxed{F^z \simeq \frac{Q^2 z}{4\pi R^3}}$$

This is what we found by images!

Another Method -- Energy

$$U = \frac{1}{2}\int_V \rho(\vec{r})\,\varphi(\vec{r})$$

$$U = \frac{1}{2}\int_V \int_{V_1} \rho(r)\,G(\vec{r},\vec{r}_1)\,\rho(\vec{r}_1)$$

Now the self interaction should be subtracted

$$U_{int} = U - U_{self} = \frac{1}{2}\int_V\int_{V_1} \rho(r)\left[G(r,r_1) - \overbrace{\frac{1}{4\pi|\vec{r}-\vec{r}_1|}}^{G_0(\vec{r},\vec{r}_1)}\right]\rho(r_1)$$

Generally we will want $G(\vec{r},\vec{r}_1) - G_0(\vec{r},\vec{r}_1) \equiv \Lambda(\vec{r},\vec{r}$

$$-\nabla^2 G(\vec{r},\vec{r}_1) = \delta^3(\vec{r}-r_1)$$

$$-\nabla^2 G_0(\vec{r},\vec{r}_1) = \delta^3(\vec{r}-\vec{r}_1)$$

The difference obeys the homogeneous eqn:

$$-\nabla^2 \Lambda(r,r_1) = 0$$