11.3 Transformation of field strengths

(a) By using the lorentz transformation rule

$$\underline{F}^{\mu\nu} = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} F^{\rho\sigma} \tag{11.82}$$

We deduced the transformation rule for the change of $F^{\rho\sigma}$ under a change of frame (boost). The $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{B}}$ fields in frame $\underline{K}$, which is moving with velocity $\boldsymbol{v}/c = \boldsymbol{\beta}$ relative to a frame K, are related to the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in frame K via

$$\underline{E}_{\parallel} = E_{\parallel} \qquad \underline{B}_{\parallel} = B_{\parallel} \tag{11.83}$$

$$\underline{\boldsymbol{E}}_{\perp} = \gamma \boldsymbol{E}_{\perp} + \gamma \boldsymbol{\beta} \times \boldsymbol{B}_{\perp} \qquad \underline{\boldsymbol{B}}_{\perp} = \gamma \boldsymbol{B}_{\perp} - \gamma \boldsymbol{\beta} \times \boldsymbol{E}_{\perp} \tag{11.84}$$

where $E_{\parallel}$ and $B_{\parallel}$ are the components of the E and B fields parallel to the boost, while $\boldsymbol{E}_{\perp}$ and $\boldsymbol{B}_{\perp}$ are the components of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields perpendicular to the boost.

(b) This is most often used to determine the magnetic field which is seen by a slow moving charge $\boldsymbol{v}/c = \boldsymbol{\beta}$, who when at rest sees only an electric field

$$\boldsymbol{B} = -\boldsymbol{\beta} \times \boldsymbol{E} \tag{11.85}$$

(c) We used this to determine the (boosted) Coulomb fields for a fast moving charge. For a charge moving along the x-axis crossing the origin $x = 0$ at time $t = 0$ the fields at longitidunal coordinate x and transverse coordinates $\boldsymbol{b} = (y, z)$

$$E_{\parallel}(t, x, \underline{)} = \frac{e}{4\pi} \frac{\gamma(x - v_p t)}{(b^2 + \gamma^2 (x - v_p t)^2)^{3/2}} \tag{11.86}$$

$$\boldsymbol{E}_{\perp}(t, x, \underline{)} = \frac{e}{4\pi} \frac{\gamma \boldsymbol{b}}{(b^2 + \gamma^2 (x - v_p t)^2)^{3/2}} \tag{11.87}$$

$$\boldsymbol{B} = \frac{\boldsymbol{v_p}}{c} \times \boldsymbol{E} \tag{11.88}$$

Note that in Eqs. 11.83, $\boldsymbol{\beta}$ is the velocity of the frame $\underline{K}$ relative to K. Thus if we know the fields in the frame of the particle (the Coulomb field), and we want to know the fields in a frame $\underline{K}$ where the particles moves with velocity $\boldsymbol{v}_p$, then $\boldsymbol{\beta} = -\boldsymbol{v}_p$ is the velocity of the frame $\underline{K}$ as seen by the particle.

11.4 Covariant actions and equations of motion

(a) Discussed the simplest of all actions

$$I[x(t)] = \underbrace{I_o}_{\text{free}} + \underbrace{I_{\text{int}}}_{\text{interaction}} \tag{11.89}$$

$$= \underbrace{\int dt\, \tfrac{1}{2} m \dot{x}^2(t)}_{\text{free}} + \underbrace{\int dt\, F_o\, x(t)}_{\text{interaction}} \tag{11.90}$$

varried this, and derived Newton's Law. All other actions follow this model.

(b) For a relativistic point particle interaction with the electromagnetic field we derived a lorentz covariant free and interation lagrangian:

i) The free part of the action is

$$I_o = -\int d\tau\, mc^2 \tag{11.91}$$

Using

$$c\, d\tau = \sqrt{-dX^\mu dX_\mu} \tag{11.92}$$

we have

$$I_o[X^\mu(p)] = -\int d\tau\, mc^2 = \int dp\, mc \sqrt{-\frac{dX^\mu}{dp}\frac{dX_\mu}{dp}} \tag{11.93}$$

We derived the equations of motion by varying this action $X^\mu(p) \to X^\mu(p) + \delta X^\mu(p)$

ii) The interaction lagrangian for a charged particle is

$$I_{\text{int}}[X^\mu(p)] = \frac{e}{c}\int dp\, \frac{dX^\mu}{dp} A_\mu(X(p)) \tag{11.94}$$

which in the non-relativistic limit reduces to

$$I_{\text{int}}[\boldsymbol{x}(t)] = \int dt\, \left[-e\varphi(t, \boldsymbol{x}(t)) + \frac{\boldsymbol{v}}{c}\cdot \boldsymbol{A}(t, \boldsymbol{x}(t))\right] \tag{11.95}$$

iii) Varying the free and interaction actions with respect to $X^\mu \to X^\mu + \delta X^\mu$

$$\delta I[X] = \delta I_o + \delta I_{\text{int}} \tag{11.96}$$

we found the equations of motion

$$m\frac{d^2 X^\mu}{d\tau^2} = eF^\mu_{\ \nu}\frac{U^\nu}{c} \tag{11.97}$$

(c) We also wrote down the action for the fields

i) The unique form invariant under Lorentz invariance, gauge invariance and parity which involves no more than two powers of the field strength is

$$I_o = \int d^4x \frac{-1}{4} F_{\mu\nu}F^{\mu\nu} \tag{11.98}$$

ii) The interaction between the currents and the fields is

$$I_{\text{int}} = \int d^4x\, J^\mu \frac{A_\mu}{c} \tag{11.99}$$

iii) Varying this action

$$\delta I = \delta I_o + \delta I_{\text{int}} \tag{11.100}$$

Yields the Maxwell equations

$$-\partial_\mu F^{\mu\nu} = \frac{J^\mu}{c} \tag{11.101}$$

iv) Demanding that the interaction part of the action I_{int} is invariant under gauge transformation leads to a requirement of current conservation:

$$\partial_\mu J^\mu = 0 \tag{11.102}$$