11 Relativity

Postulates

(a) All inertial observers have the same equations of motion and the same physical laws. Relativity explains how to translate the measurements and events according to one inertial observer to another.

(b) The speed of light is constant for all inertial frames

11.1 Elementary Relativity

Mechanics of indices, four-vectors, Lorentz transformations

(a) We desribe physics as a sequence of events labelled by their space time coordinates:

$$x^\mu = (x^0, x^1, x^2, x^3) = (c\,t, \boldsymbol{x}) \tag{11.1}$$

The space time coordinates of another inertial observer moving with velocity $\boldsymbol{v}$ relative to the first measures the coordinates of an event to be

$$\underline{x}^\mu = (\underline{x}^0, \underline{x}^1, \underline{x}^2 \underline{x}^3) = (\underline{c\,t}, \underline{\boldsymbol{x}}) \tag{11.2}$$

(b) The coordinates of an event according to the first observer x^μ determine the coordinates of an event according to another observer $\underline{x}^\mu$ through a linear change of coordinates known as a Lorentz transformation:

$$x^\mu \to \underline{x}^\mu = L^\mu_{\ \nu}(\boldsymbol{v}) x^\nu \tag{11.3}$$

I usually think of x^μ as a column vector

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{11.4}$$

so that without indices the transform

$$x \to \underline{x} = \big(L\big)\; x \tag{11.5}$$

Then to change frames from K to an observer $\underline{K}$ moving to the right with speed v relative to K the transformation matrix is

$$L^\mu_{\ \nu} = \begin{pmatrix} \gamma_v & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{11.6}$$

(c) Since the spead of light is constant for all observers we demand that

$$-(ct)^2 + \boldsymbol{x}^2 = -\underline{(ct)}^2 + \underline{\boldsymbol{x}}^2 \tag{11.7}$$

under Lorentz transformation. We also require that the set of Lorentz transformations satisfy the follow (group) requirements:

$$L(-\boldsymbol{v})L(\boldsymbol{v}) = \mathbb{I} \tag{11.8}$$

$$L(\boldsymbol{v}_2)L(\boldsymbol{v}_1) = L(\boldsymbol{v}_3) \tag{11.9}$$

here $\mathbb{I}$ is the identity matrix. These properties seem reasonable to me, since if I transform to frame moving with velocity $\boldsymbol{v}$ and then transform back to a frame moving with veloicty $-\boldsymbol{v}$, I shuld get back the same result. Similarly two Lorentz transformations produce another Lorentz transformation.

(d) Since the combination

$$-(ct)^2 + \boldsymbol{x}^2 \tag{11.10}$$

is invariant under lorentz transformation, we introduced an index notation to make such invariant forms manifest. We formalized the lowering of indices

$$x_\mu = g_{\mu\nu}x^\nu \qquad x_\mu = (-c\,t, \boldsymbol{x}) \tag{11.11}$$

with a metric tensor:

$$g_{00} = -1 \qquad g_{11} = g_{22} = g_{33} = 1 \tag{11.12}$$

In this way we define a dot product

$$x \cdot x = x^\mu x_\mu = -(ct)^2 + \boldsymbol{x}^2 \tag{11.13}$$

is manifestly invariant.

Similarly we raise indices

$$x^\mu = g^{\mu\nu}x_\nu \tag{11.14}$$

with

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{11.15}$$

Of course the process of lowering and index and then raising it agiain does nothing:

$$g^\mu_{\ \nu} = g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu_{\ \nu} = \text{identity matrix} \tag{11.16}$$

(e) Generally the upper indices are "the normal thing". We will try to leave the dimensions and name of the four vector, corresponding to that of the spatial components. Examples: $x^\mu = (ct, \boldsymbol{x})$, $A^\mu = (\varphi, \boldsymbol{A})$, $J^\mu = (c\rho, \boldsymbol{j})$, and $P^\mu = (E/c, \boldsymbol{p})$.

(f) Four vectors are anything that transforms according to the lorentz transformation $A^\mu = (A^0, \boldsymbol{A})$ like coordinates

$$\underline{A}^\mu = L^\mu_{\ \nu}A^\nu \tag{11.17}$$

Given two four vectors, A^μ and B^μ one can always construct a Lorentz invariant quantity.

$$A \cdot B = A_\mu B^\mu = -A^0B^0 + \boldsymbol{A}\cdot\boldsymbol{B} \tag{11.18}$$

(g) From the invariance of the inner prodcut we see that lower-four vectors transform with the inverse transformation and as a row,

$$x_\mu \to \underline{x}_\nu = x_\mu (L^{-1})^\mu_{\ \nu}\,. \tag{11.19}$$

I usually think of x_μ as a row

$$(x_0\ x_1\ x_2\ x_3) \tag{11.20}$$

So the transformation rule is

$$(\underline{x}_0\ \underline{x}_1\ \underline{x}_2\ \underline{x}_3) = (x_0\ x_1\ x_2\ x_3)\Big(L^{-1}\Big) \tag{11.21}$$

(h) The inverse Lorentz transform can be found by raising and lowering the indices of the transform matrix. We showed that

$$\underbrace{g_{\rho\mu}L^{\mu}{}_{\nu}g^{\nu\sigma}}_{\equiv L_{\rho}{}^{\sigma}} = (L^{-1T})_{\rho}{}^{\sigma} \tag{11.22}$$

so that if one wishes to think of a lowered four vector A_μ as a column, one has

$$\underline{A}_\nu = L_{\nu}{}^{\mu} A_\mu \tag{11.23}$$

Thus, a short excercise (done) in class shows that

$$\underline{T}^{\mu}{}_{\nu} = L^{\mu}{}_{\sigma}\, L_{\nu}{}^{\rho}\, T^{\sigma}_{\ \rho} = L^{\mu}{}_{\sigma}\, T^{\sigma}_{\ \rho}\, (L^{-1})^{\rho}_{\ \nu} \tag{11.24}$$

Doppler shift, four velocity, and propper time.

(a) The frequency and wave number form a four vector $K^\mu = (\frac{\omega}{c}, \boldsymbol{k})$. This can be used to determine a relativistic dopler shift.

(b) For a particle in motion with velocity $v_{\boldsymbol{p}}$ and gamma factor $\gamma_{\boldsymbol{p}}$, the space-time interval is

$$ds^2 = -(cdt)^2 + d\boldsymbol{x}^2\,. \tag{11.25}$$

ds^2 is associated with the clicks of the clock in the particles instantaneous rest frame, $ds^2 = -(cd\tau)^2$, so we have in any other frame

$$d\tau \equiv \sqrt{-ds^2}/c = dt\sqrt{1 - \left(\frac{dx}{dt}\right)^2/c^2} \tag{11.26}$$

$$= \frac{dt}{\gamma_{\boldsymbol{p}}} \tag{11.27}$$

(c) The four velocity of a parrticle is the distance the particle travels per propper time

$$U^\mu \equiv \frac{dx^\mu}{d\tau} = (u^0, \boldsymbol{u}) = (\gamma_p, \gamma_p \boldsymbol{v}_p) \tag{11.28}$$

so

$$\underline{U}^\mu = L^{\mu}_{\ \nu} U^\nu \tag{11.29}$$

(d) The transformation of the four velocity under lorentz transformation should be compared to the transformation of velocities. For a particle moving with velocity $\boldsymbol{v}_p$ in frame K, then in another frame $\underline{K}$ moving to the right with speed v the particle moves with velocity

$$\underline{v}_p^{\parallel} = \frac{v_p^{\parallel} - v}{1 - v_p^{\parallel} v/c^2} \tag{11.30}$$

$$\underline{v}_p^{\perp} = \frac{v_p^{\perp}}{\gamma_p(1 - v_p^{\parallel} v/c^2)} \tag{11.31}$$

where $v_p^{\parallel}$ and $v_p^{\perp}$ are the components of $\boldsymbol{v}_p$ parallel and perpendicular to v

Energy and Momentum Conservation

(a) Finally the energy and momentum form a four vector

$$P^\mu = \left(\frac{E}{c}, \boldsymbol{p}\right) \tag{11.32}$$

The invariant product of P^μ with itsself the rest energy

$$P^\mu P_\mu = -\frac{(mc^2)^2}{c^2} \tag{11.33}$$

This can be inverted giving the energy in terms of the momentum:

$$E = \sqrt{(cp)^2 + (mc^2)^2} \tag{11.34}$$

(b) Energy and Momentum are conserved in collisions, e.g. for a reaction $1 + 2 \to 3 + 4$ w have

$$P_1^\mu + P_2^\mu = P_3^\mu + P_4^\mu \tag{11.35}$$

Usually when working with collisions it makes sense to suppress c or just make the association:

$$\begin{pmatrix} E \\ p \\ m \end{pmatrix} \quad \text{is short for} \quad \begin{pmatrix} E \\ cp \\ mc^2 \end{pmatrix} \tag{11.36}$$

A starting point for analyzing the kinematics of a process is to "square" both sides with the invariant dot product $P^2 \equiv P \cdot P$. For example if $P_1 + P_2 = P_3 + P_4$ then:

$$(P_1 + P_2)^2 = (P_3 + P_4)^2 \tag{11.37}$$

$$P_1^2 + P_2^2 + 2P_1 \cdot P_2 = P_3^2 + P_4^2 + 2P_3 \cdot P_4 \tag{11.38}$$

$$-m_1^2 - m_2^2 - 2E_1E_2 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 = -m_3^2 - m_4^2 - 2E_3E_4 + 2\boldsymbol{p}_3 \cdot \boldsymbol{p}_4 \tag{11.39}$$

11.2 Covariant form of electrodynamics

(a) The players are:

i) The derivatives

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{11.40}$$

ii) The wave operator

$$\Box = \partial_\mu \partial^\mu = \frac{-1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2 \tag{11.41}$$

iii) The four velocity $U^\mu = (u^0, \boldsymbol{u}) = (\gamma_p, \gamma_p v_p)$

iv) The current four vector

$$J^\mu = (c\rho, \boldsymbol{J}) \tag{11.42}$$

v) The vector potential

$$A^\mu = (\varphi, \boldsymbol{A}) \tag{11.43}$$

vi) The field strength is a tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{11.44}$$

which ultimately comes from the relations

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\varphi \tag{11.45}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{11.46}$$

In indices we have

$$F^{0i} = E^i \qquad E^i = F^{0i} \tag{11.47}$$

$$F^{ij} = \epsilon^{ijk} B_k \qquad B_i = \tfrac{1}{2}\epsilon_{ijk} F^{jk} \tag{11.48}$$

In matrix form this anti-symmetric tensor reads

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \tag{11.49}$$

Raising and lowering indices of $F^{\mu\nu}$ can change the sign of the zero components, but does not change the ij components, *e.g.*

$$E^i = F^{0i} = -F^{i0} = F^i{}_0 = -F_0{}^i = -F_{0i} = F^0{}_i = F^{0i} \tag{11.50}$$

vii) The dual field tensor implements the replacment

$$\boldsymbol{E} \to \boldsymbol{B} \qquad \boldsymbol{B} \to -\boldsymbol{E} \tag{11.51}$$

As motivated by the maxwell equations in free space

$$\nabla \cdot \boldsymbol{E} = 0 \tag{11.52}$$

$$-\frac{1}{c}\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = 0 \tag{11.53}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{11.54}$$

$$-\frac{1}{c}\partial_t \boldsymbol{B} - \nabla \times \boldsymbol{E} = 0 \tag{11.55}$$

which are the same before and after this duality transformation. The dual field stength tensor is

$$\mathscr{F}^{\alpha\beta} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & -E^x & 0 \end{pmatrix} \tag{11.56}$$

The dual field strength tensor

$$\mathscr{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{11.57}$$

where the totally anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{even perms 0,1,2,3} \\ -1 & \text{odd perms 0,1,2,3} \\ 0 & \text{0 otherwise} \end{cases} \tag{11.58}$$

(b) The equations are

i) The continuity equation:

$$\partial_\mu J^\mu = 0 \tag{11.59}$$

$$\partial_t \rho + \nabla \cdot \boldsymbol{J} = 0 \tag{11.60}$$

ii) The wave equation in the covariant gauge

$$-\Box A^\mu = J^\mu / c \tag{11.61}$$

$$-\Box \varphi = \rho \tag{11.62}$$

$$-\Box \boldsymbol{A} = \boldsymbol{J}/c \tag{11.63}$$

This is true in the covariant gauge

$$\partial_\mu A^\mu = 0 \tag{11.64}$$

$$\frac{1}{c}\partial_t \varphi + \nabla \cdot \boldsymbol{A} = 0 \tag{11.65}$$

iii) The force law is:

$$\frac{dP^\mu}{d\tau} = eF^\mu{}_\nu \frac{U^\nu}{c} \tag{11.66}$$

$$\frac{1}{c}\frac{dE}{dt} = e\boldsymbol{E} \cdot \frac{\boldsymbol{v}}{c} \tag{11.67}$$

$$\frac{d\boldsymbol{p}}{dt} = e\boldsymbol{E} + e\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \tag{11.68}$$

If these equations are multiplied by γ they equal the relativistic equations to the left.

iv) The sourced field equations are :

$$-\partial_\mu F^{\mu\nu} = \frac{J^\nu}{c} \tag{11.69}$$

$$\nabla \cdot \boldsymbol{E} = \rho \tag{11.70}$$

$$-\frac{1}{c}\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = \frac{\boldsymbol{J}}{c} \tag{11.71}$$

v) The dual field equations are :

$$-\partial_\mu \mathscr{F}^{\mu\nu} = 0 \tag{11.72}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{11.73}$$

$$-\frac{1}{c}\partial_t \boldsymbol{B} - \nabla \times \boldsymbol{E} = 0 \tag{11.74}$$

as might have been inferred by the replacements $\boldsymbol{E} \to \boldsymbol{B}$ and $\boldsymbol{B} \to -\boldsymbol{E}$. The dual field equations can also be written in terms $F_{\mu\nu}$, and is known as the Bianchi identity:

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \tag{11.75}$$

The dual field equations are equivalent to the statement that that $F^{\mu\nu}$ can be written in terms of the gauge potential $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

vi) The conservation of energy and momentum can be written in terms of the stress tensor:

$$-\partial_\mu \Theta^{\mu\nu}_{\text{em}} = F^\mu_{\ \nu} \frac{J^\nu}{c} \tag{11.76}$$

$$-\left(\frac{1}{c}\frac{\partial u_{\text{em}}}{\partial_t} + \nabla\cdot(\boldsymbol{S}_{\text{em}}/c)\right) = \boldsymbol{E}\cdot\boldsymbol{J}/c \tag{11.77}$$

$$-\left(\frac{1}{c}\frac{\partial S^j_{\text{em}}/c}{\partial t} + \partial_i T^{ij}_{\text{M}}\right) = \rho E^j + (\boldsymbol{J}/c\times\boldsymbol{B})^j \tag{11.78}$$

Here we have defined the stress tensor

$$\Theta^{\mu\nu}_{\text{em}} = F^{\mu\lambda}F^\nu_{\ \lambda} + g^{\mu\nu}\left(-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{11.79}$$

The energy and momentum transferred from the fields $F^{\mu\nu}$ to the particles (recorded by $\Theta^{\mu\nu}_{\text{mech}}$) is

$$\partial_\mu \Theta^{\mu\nu}_{\text{mech}} = F^\mu_{\ \nu}\frac{J^\nu}{c} \tag{11.80}$$

Or

$$\partial_\mu \Theta^{\mu\nu}_{\text{mech}} + \partial_\mu \Theta^{\mu\nu}_{\text{em}} = 0 \tag{11.81}$$