1 Retarded Time and E&M fields in the radiation zone

The analysis done in class shows that:

\[ \varphi(t, r) = \frac{1}{4\pi r} \int_{r_o} \rho(T, r_o) \]  
\[ A(t, r) = \frac{1}{4\pi r} \int_{r_o} \frac{J(T, r_o)}{c} \]  

where the retarded time

\[ T = t - \frac{r}{c} + \frac{n}{c} \cdot r_o \]

Now we compute the fields

\[ B = \nabla \times A \]  
\[ E = -\frac{1}{c} \frac{\partial}{\partial t} A - \nabla \varphi \]

- We note that under the change of variables

\[ t, r_o \rightarrow T, r_o \]

the derivatives take the following form

\[ \frac{\partial}{\partial T} = \frac{\partial}{\partial t} \]  
\[ \left( \frac{\partial}{\partial r_o} \right)_T = \left( \frac{\partial}{\partial r_o} \right)_t - \frac{n}{c} \frac{\partial}{\partial t} \]

The last line should be understood as the indexed expression

\[ \left( \frac{\partial}{\partial r_o} \right)_T = \left( \frac{\partial}{\partial r_o} \right)_t - \frac{n}{c} \frac{\partial}{\partial t} \]

- We now compute \( E \) and \( B \) exploiting the derivatives in the radiation zone:

  (a) We can neglect derivatives of \( 1/r \)

\[ \frac{\partial}{\partial r} \left( \frac{1}{r^2} \right) = -\frac{n}{r^2} \]

\[ = O \left( \frac{1}{r^2} \right) \]

  (b) And we use

\[ \frac{\partial J^k}{\partial r^\ell} = \frac{\partial J^k(T, r_o)}{\partial T} \frac{\partial T}{\partial r^\ell} \]

\[ = -\frac{\partial J^k(T, r_o)}{\partial T} \frac{n}{c} + O \left( \frac{1}{r} \right) . \]

Here we have neglected the derivative \( n \) which is suppressed by \( 1/r \) relative to the leading term.

- With this we have after a bit

\[ B = -\frac{n}{c} \times \frac{1}{4\pi r} \int_{r_o} \frac{1}{c} \frac{\partial J(T, r_o)}{\partial T} \]

\[ = -\frac{n}{c} \times \frac{1}{4\pi r} \int_{r_o} \frac{1}{c} \frac{\partial J(T, r_o)}{\partial t} \]
While the E-field uses the same tricks

\[-\nabla_r \rho(T, r_o) = - \frac{\partial \rho(T, r_o)}{\partial T} \nabla_r T\]  
\[= + \frac{\partial \rho(T, r_o)}{\partial T} \frac{n}{c}\]  

(16)

(17)

to find

\[E = - \frac{1}{4\pi r^2} \int_{r_o} \frac{\partial J(T, r_o)}{\partial t} + \frac{n}{c} \frac{1}{4\pi r} \int_{r_o} \frac{\partial \rho(T, r_o)}{\partial T}\]  

(18)

Now using

\[\frac{\partial \rho(T, r_o)}{\partial T} = -(\nabla_{r_o} \cdot J)_T = -(\nabla_{r_o} \cdot J)_l + \frac{n}{c} \cdot \frac{\partial J}{\partial t}\]  

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Since \((\nabla_{r_o} \cdot J)_l\) is a total divergence, it does not contribute to the volume integral for a localized current, and we find

\[E = - \frac{1}{4\pi c^2} \int_{r_o} \left[ \partial_t J - n(n \cdot \partial_t J) \right] \]  

the part of \(\partial_t J\) transverse to \(n\)

(20)

• To see that the electric field is orthogonal to \(B\) we use that the transverse components of a vector \(V\):

\[V - n(n \cdot V) = -n \times (n \times V)\]  

(21)

Leading this to

\[E = n \times \left[ \frac{n}{c} \times \frac{1}{4\pi r} \int_{r_o} \frac{1}{c} \frac{\partial J(T, r_o)}{\partial t} \right]\]  

(22)

\[= - n \times B\]  

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