2.3 Solving the Laplace Equation by Separation: Lecture 3,4,5

A summary of separation of variables in different coordinate systems is given in Appendix **??** and Appendix A.2.

Solving the Laplace equation: Chapter 7

Sec 7.1 to 7.9: We use a technique of separation of variables in different coordinate systems. The technique of separation of variables is best illustrated by example. For instance consider a potential in a cylindrical

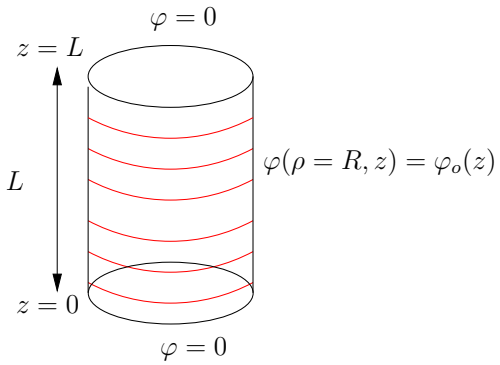

Figure 2.1: A cylinder used to illustrate separation of vars

geometry. The potential $\varphi(\rho, z)$ is specified at a given radius R to be $\varphi_o(z)$.

(a) We look for solutions of the separated form

$$\varphi = \underbrace{R(\rho)}_{\perp \text{ to surf}} \quad \underbrace{Z(z)}_{\parallel \text{ to surf}} \tag{2.38}$$

Leading to the two equations

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + k^2\right] R_k(\rho) = 0 \tag{2.39}$$

$$\frac{-\partial^2 Z_k}{\partial z^2} = k^2 Z_k \tag{2.40}$$

(b) It is best to analyze the parallel equations first which are all of the form of a Sturm Louiville *eigenvalue* equation, determining the eigen-functions Z_k *and* the eigenvalues (or separation constants) k. For the problem at hand

$$Z_k = A_k \cos kz + B_k \sin kz \tag{2.41}$$

is the general solution. In order to satisfy the boundary condition $Z_k(0) = Z_k(L) = 0$, we must have $A_k = 0$ and $k = n\pi/L$, leading to

$$Z_k = B_n \sin(k_n z) \qquad k_n = \frac{n\pi}{L} \quad n = 1, 2, \ldots \tag{2.42}$$

Thus the parallel directions determine both the functions and the separation constants

(c) The perpendicular equations are solved with a specified k. These equations do not usually constrain the separation constant. The general solution is

$$R_k(\rho) = AI_0(k\rho) + BK_0(k\rho) \tag{2.43}$$

(d) Finally the generall solution is a sum over the eigen-functions times the perpedicular solution

$$\varphi = \sum_n \sin k_n z \left[A_k I_0(k_n \rho) + B_k K_0(k_n \rho)\right] \tag{2.44}$$

Since the eigen-fcns are complete and orthogonal we will a always be able to adjust the constants A_k and B_k to obtain the boundary condition at $\rho = R$

Solving the separated equations

After separating variables, *all* of the equations we wil study can be written in Sturm Louiville form:

$$\left[\frac{-d}{dx}p(x)\frac{d}{dx} + q(x)\right] y(x) = \lambda r(x) y(x) \tag{2.45}$$

where $p(x)$ and $r(x)$ are postive definite fcns.

(a) If boundary conditions are specified at endpoints a and b then the problem becomes an eigen-value equation. Only certain values of λ_n are allowed and the functions are uniquely determined up to a constant

$$\left[\frac{-d}{dx}p(x)\frac{d}{dx} + q(x)\right] \psi_n(x) = \lambda_n r(x) \psi_n(x) \tag{2.46}$$

The parallel equations will have this form, see Eq. (2.41) and Eq. (2.42) and notice how the boundary conditions at $a = 0$ and $b = L$ fixed the value of k_n.

(b) The resulting eigenfunctions are complete and orthogonal with respect to the weight $r(x)$

$$\int_a^b dx\, r(x) \psi_n^*(x) \psi_m(x) = 0 \qquad n \neq m \tag{2.47}$$

where a and b are the endpoints where the boundary conditions are specified.

(c) Given two independent solutions to the differential equation $y_1(x)$ a and $y_2(x)$ (not necessarily eigen-fcns which since e-fcns also satisfy the boundary conditions at a and b), The wronskian times $p(x)$ is constant.

$$p(x)\left[y_1(x) y_2'(x) - y_2(x) y_1'(x)\right] = const \tag{2.48}$$

This usually amounts to a statement of Gauss Law. For Bessels equation this means that

$$k\rho\left[I_0(k\rho) K_0'(k\rho) - K_0(k\rho) I_0'(k\rho)\right] = const \tag{2.49}$$

Solving the separated equations with δ function source terms

(a) We will also need to know the green function of the one dimensional equation

$$\left[\frac{-d}{dx}p(x)\frac{d}{dx} + q(x)\right] g(x, x_o) = \delta(x - x_o) \tag{2.50}$$

The Green function for such 1D equations is based on knowing two homogeneous solutions $y_{out}(x)$ and $y_{in}(x)$, where $y_{out}(x)$ satisfies the boundary conditions for $x > x_o$, and $y_{in}(x)$ satisfies the boundary conditions for $x < x_o$.

The Green function is continuous but has discontinuous derivatives. Since we know the solutions outside and inside it takes the form:

$$\begin{aligned} G(x, x_o) =& C\left[y_{out}(x) y_{in}(x_o)\theta(x - x_o) + y_{in}(x) y_{out}(x_o)\theta(x_o - x)\right] && (2.51)\\ \equiv& C y_{out}(x_>) y_{in}(x_<) && (2.52) \end{aligned}$$

where C is a constant determined by integrating the equation, Eq. (2.50), across the delta function. In the second line we use the common (but somewhat confusing notation)

$$x_> \equiv \text{the greater of } x \text{ and } x_o \tag{2.53}$$

$$x_< \equiv \text{the smaller of } x \text{ and } x_o \tag{2.54}$$

which makes the second line mean the same as the first line.

Integrating from $x = x_o - \epsilon$ to $x = x_o + \epsilon$ we find the jump condition which enters in many problems:

$$-p(x)\frac{dg}{dx}\bigg|_{x_o+\epsilon} + p(x)\frac{dg}{dx}\bigg|_{x_o-\epsilon} = 1\,, \tag{2.55}$$

which can be used to find C.

(b) In fact the jump condition will always involve the Wronskian of the two solutions. Substituting Eq. (2.51) into Eq. (2.55) we see that $C = 1/(p(x_o)W(x_o))$

$$G(x,x_o) = \frac{[y_{out}(x)y_{in}(x_o)\theta(x-x_o) + y_{in}(x)y_{out}(x_o)\theta(x_o-x)]}{p(x_o)W(x_o)} \tag{2.56}$$

$$\equiv \frac{y_{out}(x_>)y_{in}(x_<)}{p(x_o)W(x_o)} \tag{2.57}$$

where $W(x_o) = y_{out}(x_o)y'_{in}(x_o) - y_{in}(x_o)y'_{out}(x_o)$ is the Wronskian. Note that the denominator $p(x_o)W(x_o)$ is constant and is independent of x_o.

2.4 Green functions and the Poisson equation: Lectures 4,6,8,9:

Green Functions and the Poisson equation: Chapter 8

(a) sec 8.4 The Dirichlet Green function satisfies the Poisson equation with delta-function charge

$$-\nabla^2 G_D(\boldsymbol{r}, \boldsymbol{r}_o) = \delta^3(\boldsymbol{r} - \boldsymbol{r}_o) \tag{2.58}$$

and vanishes on the boundary. *It is the potential at* $\boldsymbol{r}$ *due to a point charge (with unit charge) at* $\boldsymbol{r}_o$ The simplest free space green function is just the point charge solution

$$G_D = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{2.59}$$

In two dimensions the Green function is

$$G_D = \frac{-1}{2\pi} \log|\boldsymbol{r} - \boldsymbol{r}_o| \tag{2.60}$$

which is the potential from a line of charge with charge density $\lambda = 1$

(b) sec 8.4.1. The Poisson equation or the boundary value problem of the Laplace equation can be solved once the Dirichlet Green function is known. By examining the Wronskian of the Green function and the solution of interest we showed that

$$\varphi(\boldsymbol{r}) = \int_V \mathrm{d}^3 r_o \, G_D(\boldsymbol{r}, \boldsymbol{r}_o)\rho(\boldsymbol{r}_o) - \int dS_o \, \boldsymbol{n}_o \cdot \nabla_{r_o} G_D(\boldsymbol{r}, \boldsymbol{r}_o)\varphi(r_o) \tag{2.61}$$

where $\boldsymbol{n}_o$ is the *outward* normal

(c) sec: 8.3. A useful technique to find a Green function is image charges. You should know the image charge green functions

i) A plane in 1D and 2D (class)

ii) A sphere (homework)

iii) A cylinder (homework + recitation)

(d) sec: 8.5.3: one technique to find the green function is to expand the $\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$ in eigenfunctions. For a complete set of normalized eigen functions of the Laplace operator satisfying the boundary conditions, i.e.

$$-\nabla^2 \psi_n = \lambda_n \psi_n \tag{2.62}$$

and

$$\sum_n \psi_n(\boldsymbol{r})\psi_n^*(\boldsymbol{r}_o) = \delta^3(\boldsymbol{r} - \boldsymbol{r}_o) \tag{2.63}$$

The Green fcn can be written:

$$G_D = \sum_n \frac{\psi_n(\boldsymbol{r})\psi_n^*(\boldsymbol{r}_o)}{\lambda_n} \tag{2.64}$$

The primary use of this type of expansion is to explain eqs. like

$$\frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r} - \boldsymbol{r}_o)}}{k^2} \tag{2.65}$$

(e) sec: 8.5.4 and 8.5.5, method of direct integration: This is best illustrated by example. Pick two dimensions of a surface (say θ, ϕ). The method is motivated by the fact that $\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$ can be written as a sum

$$\delta^3(\boldsymbol{r} - \boldsymbol{r}_o) = \frac{1}{r^2}\delta(r - r_o)\delta(\cos\theta - \cos\theta_o)\delta(\phi - \phi_o) = \frac{1}{r^2}\delta(r - r_o)\sum_{\ell m} Y_{\ell m}(\theta, \phi)Y_{\ell m}^*(\theta_o, \phi_o) \tag{2.66}$$

Thus the green function is can also be written as

$$G(\boldsymbol{r},\boldsymbol{r}_o)=\sum_{\ell=0}^{\infty}\sum_{-\ell}^{\ell}g_{\ell m}(r,r_o)Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta_o,\phi_o) \tag{2.67}$$

leading to an equation for $g_{\ell m}(r,r_o)$

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{\ell(\ell+1)}{r^2}\right]g_{\ell m}(r,r_o)=\frac{1}{r^2}\delta(r-r_o) \tag{2.68}$$

This remaining equation in 1D is then solved for the green function following the strategy outlined above in Sect. **??** (see Eq. (2.50)). This depends on the boundary conditions.

Similar expressions can be derived in other coordinates. For instance, using the result in cylindrical coords

$$\delta^3(\boldsymbol{r}-\boldsymbol{r}_o)=\frac{1}{\rho}\delta(\rho-\rho_o)\delta(z-z_o)\delta(\phi-\phi_o)=\frac{1}{\rho}\delta(\rho-\rho_o)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi}\left[e^{im(\phi-\phi_o)}e^{i\kappa(z-z_o)}\right] \tag{2.69}$$

Leading to

$$G(\boldsymbol{r},\boldsymbol{r}_o)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi}\left[e^{im(\phi-\phi_o)}e^{i\kappa(z-z_o}\right]g_{m\kappa}(\rho,\rho_o) \tag{2.70}$$

where

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\kappa^2+\frac{m^2}{\rho^2}\right]g_{m\kappa}(\rho,\rho_o)=\frac{1}{\rho}\delta(\rho-\rho_o) \tag{2.71}$$

(f) sec 8.5.4: For free space, the two solutions to Eq. (2.68) are $y_{out}(r)=1/r^{\ell+1}$ and $y_{in}(r)=r^{\ell}$. Then the free space Green fcn can be written

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}=\sum_{\ell=0}^{\infty}\sum_{-\ell}^{\ell}\left[Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta_o,\phi_o)\right]\frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}} \tag{2.72}$$

Some useful identities can be derived from Eq. (2.72):

i) The generating function of Legendre Polynomials is found by setting $\boldsymbol{r}_o=\hat{\boldsymbol{z}}$ and $r<1$ with $Y_{\ell 0}=\sqrt{(2\ell+1)/4\pi}P_\ell(\cos\theta)$

$$\frac{1}{\sqrt{1+r^2-2r\cos\theta}}=\sum_{\ell=0}^{\infty}r^{\ell}P_\ell(\cos\theta) \tag{2.73}$$

ii) The spherical harmonic addition theorem which we find by writing by setting $\boldsymbol{r}_o=1$ and $r<1$ and using $1/|\boldsymbol{r}-\boldsymbol{r}_o|=1/\sqrt{1+r^2-2r\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}_o}$

$$P_\ell(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}_o)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta_o\phi_o) \tag{2.74}$$

where $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}_o$ is the cosine of the angle between the two vectors.

iii) The shell structure relation which you find by setting $\hat{\boldsymbol{r}}=\hat{\boldsymbol{r}}_o$

$$1=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta,\phi) \tag{2.75}$$

This relation is what is responsible for shell structure in the periodic table

(g) Similar expansion exists in other coordinates. *e.g.* in cylindrical coords $y_{out}(\rho)=K_m(\kappa\rho)$ and $y_{in}(\rho)=I_m(\kappa\rho)$, leading to

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int\frac{dk}{2\pi}\left[e^{im(\phi-\phi_o)}e^{ik(z-z_o)}\right]I_m(k\rho_<)K_m(k\rho_>) \tag{2.76}$$