4 Ohms Law and Conduction

4.1 Steady current and Ohms Law: Lecture 17

(a) For steady currents

$$\nabla \cdot \boldsymbol{j} = 0 \tag{4.1}$$

(b) For steady currents in ohmic matter

$$\boldsymbol{j} = \sigma \boldsymbol{E} \tag{4.2}$$

(c) σ has units of $1/s$. Note that in MKS units σ_{MKS} has the uninformative unit $1/\mathrm{o}hm\, m$:

$$\sigma_{HL} = \frac{\sigma_{MKS}}{\varepsilon_o} \tag{4.3}$$

For $\sigma_{MKS} = 10^7 1/(\mathrm{o}hm\, m)$ we find $\sigma \sim 10^{18} 1/s$.

(d) To find the flow of current we need to solve the electrostatics problem

$$-\nabla \cdot (\sigma \boldsymbol{E}) = 0 \tag{4.4}$$

$$\nabla \times \boldsymbol{E} = 0 \tag{4.5}$$

or for homogeneous material

$$-\sigma \nabla^2 \varphi = 0 \tag{4.6}$$

We see that we are supposed to solve the Laplace equation. However the boundary conditions are rather different.

(e) A point source of current is represented by a delta function $I\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$. While a sink of current is represented by a delta function of opposite sign $-I\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)$.

(f) Eq. (4.4) and Eq. (4.6) need boundary conditions. At an interface current should be conserved so

$$\boldsymbol{n} \cdot (\boldsymbol{j}_2 - \boldsymbol{j}_1) = 0 \tag{4.7}$$

or

$$\sigma_2 \frac{\partial \varphi_2}{\partial n} = \sigma_1 \frac{\partial \varphi_1}{\partial n} \tag{4.8}$$

Most often this is used to say that the normal component of the Electric field at a metal-insulator interface should be zero:

$$\boldsymbol{n} \cdot \boldsymbol{E} = 0 \qquad \text{at metal-insulator interface} \tag{4.9}$$

(g) In general the input current (or normal derivatives of the potential) must be specified at all the boundaries in order to have a well posed boundary value problem that can be solved (at least numerically.)

(h) In general the input currents $I_a = I_1, I_2, \ldots$ on a set conductors will be will be specified, specifying the normal derivatives on all of the surfaces. Then you solve for the potential. The voltages of a given electrode relative to ground is V_a , and you will find that $V_a = \sum_b R_{ab} I_b$. R_{ab} is the resistance matrix.

4.2 Basic physics of metals, Drude model of conductivity: Lecture 22

This section really lies outside of electrodynamics. But it helps to understand what is going on.

(a) The electrons in the metal under go scatterings with impurities and other defects on a time scale τ_c. For copper:

$$\tau_c \sim 10^{-14} s \tag{4.10}$$

(b) A typical coulomb oscillation / orbital frequency is set by the plasma frequency

$$\omega_p = \sqrt{\frac{ne^2}{m}} \tag{4.11}$$

For copper $\omega_{\boldsymbol{p}}$ is of order a typical quantum frequency and scales like:

$$\omega_p \sim \Big(\frac{1}{m} \underbrace{\frac{e^2}{a_o^3 m}}_{\text{spring const}} \Big)^{1/2} \tag{4.12}$$

$$\sim \left(\frac{27.2\,\mathrm{e}V}{\hbar}\right) \tag{4.13}$$

$$\sim 10^{-16}\ 1/\mathrm{s} \tag{4.14}$$

In the second to last line we ignored all 4π factors and used Bohr model identities

$$\frac{1}{2}\left(\frac{e^2}{4\pi a_o}\right) = \frac{\hbar^2}{2ma_o^2} = 13.6\,\mathrm{e}V \tag{4.15}$$

which you can remember by noting that (minus) coulomb potential energy is twice the kinetic energy=$p^2/2m$ and knowing $p_{bohr} = \hbar/a_o$ as expected by the uncertainty principle.

(c) Since the distances between collisions are long compared to the Debroglie wavelength, and the time between collisions is long compared to a typical inverse quantum frequency, we are justified in using classical transport

$$\omega_p \tau_c \sim 100 \gg 1 \tag{4.16}$$

(d) In the Drude model the magnitude of the driving force $F_E = eE_{\mathrm{ext}}$ equals the magnitude drag force $F_{\mathrm{d}rag} = m\boldsymbol{v}/\tau_c$, leading to an estimate of the conductivity

$$\sigma = \frac{ne^2\tau_c}{m} = \omega_p^2\,\tau_c \tag{4.17}$$

The estimates given show

$$\sigma \sim 10^{18}\ \mathrm{s}^{-1} \tag{4.18}$$

for a metal like copper.

5 Magneto Statics and Magnetic Matter

5.1 Magneto-Statics: Lectures 14, 15, 16

At first order in $1/c$ we have the magneto static equations

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}_{tot}}{c} \qquad \boldsymbol{j}_{\rm tot} = \frac{\boldsymbol{j}}{c} + \underbrace{\frac{1}{c}\partial_t \boldsymbol{E}^{(0)}}_{\text{displacement current}} \tag{5.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{5.2}$$

where $\boldsymbol{j}_D = 1/c\, \partial_t \boldsymbol{E}^{(0)}$ is the displacement current. The formulas given below assume that $\boldsymbol{j}_D$ is zero. But, with no exceptions apply if one replaces $\boldsymbol{j} \to \boldsymbol{j} + \boldsymbol{j}_D$.

The current is taken to be steady

$$\nabla \cdot \boldsymbol{j} = 0 \tag{5.3}$$

Computing Fields: Lecture 14 and 15

(a) Below we note that for a current carrying wire

$$\boldsymbol{j}\mathrm{d}^3 r = I \mathrm{d}\boldsymbol{\ell} \tag{5.4}$$

(b) We can compute the fields using the integral form of Ampères law $\nabla \times \boldsymbol{B} = j/c$, which says that the loop integral of $\boldsymbol{B}$ is equal to the current piercing the area bounded by the loop

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{I_{\rm pierce}}{c} \tag{5.5}$$

For the familiar case of a current carrying wire we found $B_\phi = (I/c)/2\pi\rho$, where ρ is the distance from the wire.

(c) The Biot-Savat Law is seemingly similar to the coulomb law

$$\boldsymbol{B}(\boldsymbol{r}) = \int \mathrm{d}^3 r_o \, \frac{\boldsymbol{j}(\boldsymbol{r}_o)/c \times \widehat{\boldsymbol{r} - \boldsymbol{r}_o}}{4\pi |\boldsymbol{r} - \boldsymbol{r}_o|^2} \tag{5.6}$$

We used this to compute the magnetic field of a ring of radius on the z-axis

$$B_z = 2\frac{(I/c)\pi a^2}{4\pi\sqrt{z^2 + a^2}} \tag{5.7}$$

which you can remember by knowing magnetic moment of the ring and other facts about magnetic dipoles (see below)

(d) Using the fact that $\nabla \cdot \boldsymbol{B} = 0$ we can write it as the curl of $\boldsymbol{A}$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda \tag{5.8}$$

but recognize that we can always add a gradient of a scalar function Λ to $\boldsymbol{A}$ without changing $\boldsymbol{B}$.

(e) If we adopt the coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ and use the much used identity

$$\nabla \times (\nabla \times A) = -\nabla^2 \boldsymbol{A} + \boldsymbol{\nabla}(\nabla \cdot \boldsymbol{A}), \tag{5.9}$$

we get the result

$$-\nabla^2 \boldsymbol{A} = \frac{\boldsymbol{j}}{c}. \tag{5.10}$$

Then in free space $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}(\boldsymbol{r}) = \int d^3 r_o \frac{\boldsymbol{j}(\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{5.11}$$

(f) The equations must be supplemented by boundary conditions. In vacuum we have that the parallel components of $\boldsymbol{B}$ jump according to size of the surface currents $\boldsymbol{K}$, while the normal components of $\boldsymbol{B}$ are continuous

$$\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}}{c} \tag{5.12}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{5.13}$$

Multipole expansion of magnetic fields: Lecture 16

We wish to compute the magnetic field far from a localized set of currents. We can start with Eq. (5.14) and determine that far from the sources the vector potential is described by the magnetic dipole moment:

(a) The vector potential is

$$\boldsymbol{A} = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{5.14}$$

where

$$\boldsymbol{m} = \tfrac{1}{2} \int d^3 r_o \boldsymbol{r}_o \times \boldsymbol{j}(\boldsymbol{r}_o)/c \tag{5.15}$$

is the magnetic dipole moment.

(b) For a current carrying wire:

$$\boldsymbol{m} = \frac{I}{c} \frac{1}{2} \oint \boldsymbol{r}_o \times d\boldsymbol{\ell}_o = \frac{I}{c} \boldsymbol{a} \tag{5.16}$$

(c) The magnetic field from a dipole

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{3(\boldsymbol{n} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3} \tag{5.17}$$

(d) **U**NITS NOTE: I defined $\boldsymbol{m}$ in Eq. (5.15) with $\boldsymbol{j}/c$. This has the "feature" that that

$$\boldsymbol{m}_{HL} = \frac{\boldsymbol{m}_{MKS}}{c} \tag{5.18}$$

In MKS units

$$\boldsymbol{A}_{MKS} = \mu_o \frac{\boldsymbol{m}_{MKS} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{5.19}$$

Setting $\varepsilon_o = 1$ so $\mu_o = 1/c^2$ and multiplying by c

$$\boldsymbol{A}_{HL} = c\boldsymbol{A}_{MKS} = \frac{\boldsymbol{m}_{MKS}/c \times \hat{\boldsymbol{r}}}{4\pi r^2} = \frac{\boldsymbol{m}_{HL} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{5.20}$$

Below we will define the magnetization, and similarly $\boldsymbol{M}_{HL} = \boldsymbol{M}_{MKS}/c$.

Separation of variables with magnetic problems

There are two cases where the equations for $\boldsymbol{A}$ simplify.

(a) If the current is azimuthally symmetric then it is reasonable to try a form $A_\phi(r,\theta)$

$$-\nabla^2 \boldsymbol{A} = \mu \frac{\boldsymbol{j}}{c} \Rightarrow -\nabla^2 A_\phi + \frac{A_\phi}{r^2 \sin^2\theta} = \mu \frac{j_\phi}{c} \tag{5.21}$$

This is similar to the method of solution presented in

(b) If the current runs up and down then you can try $A_z(\rho,\phi)$ in cylindrical coordinates:

$$-\nabla^2 A_z(\rho,\phi) = \mu \frac{j_z}{c} \tag{5.22}$$

Forces on currents: Lecture 16

(a) We wish to compute the force on a small current carrying object in an external magnetic field. For a compact region of current (which is small compared to the inverse gradients of the external magnetic field) the total magnetic force is

$$\boldsymbol{F}(\boldsymbol{r}_o) = (\boldsymbol{m}\cdot\nabla)\,\boldsymbol{B}(\boldsymbol{r}_o) \tag{5.23}$$

where $\boldsymbol{m}$ is measured with respect $\boldsymbol{r}_o$, *i.e.*

$$\boldsymbol{m} = \frac{1}{2}\int_V \mathrm{d}^3r\ \delta\boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r})/c \tag{5.24}$$

with $\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_o$.

(b) For a fixed dipole magnitude we have $\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B})$ or

$$U(\boldsymbol{r}_o) = -\boldsymbol{m}\cdot\boldsymbol{B}(\boldsymbol{r}_o) \tag{5.25}$$

This formula is the same as the MKS one since we have taken $\boldsymbol{m}_{HL} = \boldsymbol{m}_{MKS}/c$.

(c) The torque is

$$\boldsymbol{\tau} = \boldsymbol{m}\times\boldsymbol{B} \tag{5.26}$$

(d) Finally (we included this later) the magnetic force on a current carrying region is

$$(\boldsymbol{F}_B)^j = \frac{1}{c}\int_V (\boldsymbol{j}\times\boldsymbol{B})^j = -\int_{\partial V} dS\,\boldsymbol{n}_i T_B^{ij} \tag{5.27}$$

where

$$T_B^{ij} = -B^iB^j + \frac{1}{2}\boldsymbol{B}^2\delta^{ij} \tag{5.28}$$

is the magnetic stress tensor and $\boldsymbol{n}$ is an outward directed normal.