5.2 Magnetic Matter: Lectures 18, 19

Basic equations

(a) We are considering materials in the presence of a magnetic field. We write $\boldsymbol{j}$ as an expansion in terms of the derivatives in the magnetic field. For weak fields, and an isotropic medium , the lowest term in the derivative expansion, for a parity and time-reversal invariant material is

$$\boldsymbol{j}_b = c\chi_m^B \nabla \times \boldsymbol{B} \tag{5.29}$$

where we have inserted a factor of c for later convenience.

(b) The current takes the form

$$\boldsymbol{j}_b = c\nabla \times \boldsymbol{M} \tag{5.30}$$

- $\boldsymbol{M}$ is known as the magnetization, and can be interpreted as the magnetic dipole moment per volume.
- We have worked with linear response for an isotropic medium where

$$\boldsymbol{M} = \chi_m^B \boldsymbol{B} \tag{5.31}$$

  This is most often what we will assume.
- Usually people work with $\boldsymbol{H}$ (see the next item for the definition of $\boldsymbol{H}$) not $\boldsymbol{B}$ [1]

$$\boldsymbol{M} = \chi_m \boldsymbol{H} \tag{5.32}$$

- For not-that soft ferromagnets $\boldsymbol{M}(\boldsymbol{B})$ can be a very non-linear function of $\boldsymbol{B}$. This will need to be specified (usually by experiment) before any problems can be solved. Usually this is expressed as the magnetic field as a function of $\boldsymbol{H}$

$$\boldsymbol{B}(\boldsymbol{H}) \tag{5.33}$$

  where $\boldsymbol{H}$ is small (of order gauss) and $\boldsymbol{B}$ is large (of order Tesla)

(c) After specifying the currents in matter, Maxwell equations take the form

$$\nabla \times \boldsymbol{B} = \nabla \times \boldsymbol{M} + \frac{\boldsymbol{j}_{\rm ext}}{c} \tag{5.34}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{5.35}$$

or

$$\nabla \times \boldsymbol{H} = \frac{\boldsymbol{j}_{\rm ext}}{c} \tag{5.36}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{5.37}$$

where [2]

$$\boldsymbol{H} = \boldsymbol{B} - \boldsymbol{M} \tag{5.39}$$

(d) For linear materials :

$$\boldsymbol{B} = \mu \boldsymbol{H} = \frac{1}{1-\chi_m^B}\boldsymbol{H} = (1+\chi_m)\boldsymbol{H} \tag{5.40}$$

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[1] There are a couple of reasons for this. One reason is because the parallel components of H are continuous across the sample. But, ultimately it is $\boldsymbol{B}$ which is the curl $\boldsymbol{A}$, and it is ultimately the average current which responds to the gauge potential, through a retarded medium current-current correlation function that we wish to categorize.

[2] In the MKS system one has $\boldsymbol{H}_{MKS} = \frac{1}{\mu_o}\boldsymbol{B}_{MKS} - M_{MKS}$ so that $\boldsymbol{B}$ and $\boldsymbol{H}$ have different units. In a system of units where $\varepsilon_o = 1$ (so $1/\mu_o = c^2$) we have $H_{HL} = H_{MKS}/c$, $\boldsymbol{M}_{HL} = \boldsymbol{M}_{MKS}/c$ or (since $1/c = \sqrt{\mu_o}$)

$$\boldsymbol{H}_{HL} = \sqrt{\mu_o} H_{MKS} \qquad \boldsymbol{M}_{HL} = \sqrt{\mu_o}\boldsymbol{M}_{MKS} \tag{5.38}$$

Solving magnetostatic problems with media:

(a) For linear materials in the coulomb gauge we get

$$\nabla \times \boldsymbol{H} = \mu \frac{\boldsymbol{j}_{ext}}{c} \tag{5.41}$$

$$\nabla \cdot \boldsymbol{H} = 0 \tag{5.42}$$

and with $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

$$-\nabla^2 \boldsymbol{A} = \mu \frac{j_{ext}}{c} \tag{5.43}$$

which can be solved using the methods of magnetostatics.

(b) To solve magneto static equations we have boundary conditions:

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \frac{\boldsymbol{K}_{ext}}{c} \tag{5.44}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{5.45}$$

i.e. if there are no external currents then the parallel components of $\boldsymbol{H}$ are continuous and the perpendicular components of $\boldsymbol{B}$ are continuous.

(c) At an interface the there are bound currents which are generated

$$\boldsymbol{n} \times (\boldsymbol{M}_2 - \boldsymbol{M}_1) = \frac{\boldsymbol{K}_b}{c} \tag{5.46}$$

(d) In a hard ferromagnet $\boldsymbol{M}(\boldsymbol{r}_o)$ is specified and we solve the equations :

$$\nabla \times \boldsymbol{B} = \nabla \times \boldsymbol{M} + \frac{\boldsymbol{j}_{ext}}{c} \qquad \nabla \times \boldsymbol{H} = \frac{\boldsymbol{j}_{ext}}{c} \tag{5.47}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M} \tag{5.48}$$

This can be done by introducing a vector potential and solving for $\boldsymbol{A}$ (as done in class), or if there are only surface currents by introducing the magnetic scalar potential, ψ_m; solving for the magnetic scalar potential in each (current-free) region where $\nabla \times \boldsymbol{H} = 0$; and matching the ψ_m in each region across the interfaces, with the boundary conditions. In this case $-\nabla \cdot \boldsymbol{M}$ acts as a magnetic charge in each region

$$-\nabla^2 \psi_m = -\nabla \cdot \boldsymbol{M} \tag{5.49}$$

Magnetic Materials

We divide magnetic materials into Ferromagnetic substances and diamagnetic and paramagnetic magnetic substances.

(a) First consider Eq. (5.29). Dimensional analysis together with the recognition that magnetic forces, $F_B = q(\boldsymbol{v}/c) \times \boldsymbol{B}$, are smaller than electric forces by a factor of v/c, leads to an estimate that

$$\chi_m^B \sim \frac{v^2}{c^2} \tag{5.50}$$

where v is a typical speed of an electron in material.

For a typical electron speed $v/c \sim \alpha \sim 1/137$ we find that

$$\chi_m^B \sim 10^{-5}\,. \tag{5.51}$$

This is about the right for diamagnetic and paramagnetic substances where the magnetic response is small, but totally wrong for Ferro-magnetic substances.

(b) In diamagnetic substances, $\chi_m^B < 0$, and $\mu < 1$. In this case the current in Eq. (5.29) arises as a magnetic interaction between the external field B and the orbital angular momentum. This is the dominant case source of interactions in atoms with no electrons in the outer shell.

(c) In paramagnetism, $\chi_m^B > 0$ and the $\mu > 1$. This typically arises as the spins tend to align with the applied magnetic field. Thus it arises in systems with unpaired spins.

(d) In strongly ferromagnetic substances, the exchange interactions between the electrons tend to align the spins, even in the absence of an applied field. The origin of this interaction is electrostatic, *i.e.* the spatial part of the electrons wave function can be further apart if if it is in anti-symmetric state. This makes the spin part of the wave function be symmetric (*i.e.* aligned.). The magnetic spins work cooperatively to form large domains of aligned spins. The domain structure arises because the large magnetic fields produced by the *many* aligned spins is competing with the short range coulomb forces. The applied magnetic field tends to align the domains, leading to a much large magnetic response than is estimated from Eq. (5.51)