6 A Summary of Maxwell Equations

6.1 The Maxwell Equations a Summary : Lecture 21

The maxwell equations in linear media can be written down for the gauge potentials. You should feel comfortable deriving all of these results directly from the Maxwell equations:

(a) The fields are

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{6.1}$$

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\varphi \tag{6.2}$$

(b) The equations of motion for the gauge potentials are in any gauge

$$-\Box\varphi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot\boldsymbol{A}\right) = \rho \tag{6.3}$$

$$-\Box\boldsymbol{A} + \boldsymbol{\nabla}\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot\boldsymbol{A}\right) = \frac{\boldsymbol{j}}{c} \tag{6.4}$$

where the d'Alembertian is

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \tag{6.5}$$

Note that these equations for φ and $\boldsymbol{A}$ can not be solved without specifying a gauge constraint, *i.*e. given current conservation:

$$\partial_t\rho + \nabla\cdot\boldsymbol{j} = 0 \tag{6.6}$$

There are actually only three equations, but four unknowns.

(c) If the *coulomb gauge* is specified

$$\nabla\cdot\boldsymbol{A} = 0 \tag{6.7}$$

the equations read:

$$-\nabla^2\varphi = \rho \tag{6.8}$$

$$-\Box\boldsymbol{A} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t(-\nabla\varphi) \tag{6.9}$$

(d) If the *covariant gauge* is specified

$$\frac{1}{c}\partial_t\varphi + \nabla\cdot\boldsymbol{A} = 0 \tag{6.10}$$

then the equations read

$$-\Box\varphi = \rho \tag{6.11}$$

$$-\Box\boldsymbol{A} = \frac{\boldsymbol{j}}{c} \tag{6.12}$$

7 Induction and Quasi-Static Fields

7.1 Induction and the energy in static Magnetic fields: Lecture 20

(a) The Faraday law of induction says that changing magnetic flux induces an electric field

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B} \tag{7.1}$$

In integral form

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{1}{c}\partial_t \Phi_B \qquad \Phi_B = \int_{\partial V} \boldsymbol{B} \cdot \mathrm{d}\mathbf{a} \tag{7.2}$$

(b) Faraday's Law is suppressed by $1/c^2$ relative to the coulomb law

(c) Faraday's law can be used to compute the energy stored in a magneto static field. As the currents are increased and the magnetic field is changed, the increase in energy stored in the magnetic fields and associated magnetization is

$$\delta U = \int_V \boldsymbol{H} \cdot \delta \boldsymbol{B}\, dV \tag{7.3}$$

For linear material $\boldsymbol{B} = \mu \boldsymbol{H}$

$$U = \frac{1}{2}\int \boldsymbol{H} \cdot \boldsymbol{B}\, \mathrm{d}^3 r \tag{7.4}$$

$$= \frac{1}{2\mu}\int \boldsymbol{B} \cdot \boldsymbol{B}\, \mathrm{d}^3 r \tag{7.5}$$

This can also be expressed in terms of $\boldsymbol{A}$:

$$\delta U = \int_V \frac{\boldsymbol{j}}{c} \cdot \delta \boldsymbol{A} \tag{7.6}$$

and for linear material:

$$U = \frac{1}{2}\int_V \frac{\boldsymbol{j}}{c} \cdot \boldsymbol{A} \tag{7.7}$$

The factor 1/2 arises because we are double counting the integral over the current in much the same way that a factor of 1/2 appears in $U = \frac{1}{2}\int_V \rho\varphi$

(d) Using the coulomb gauge result, for vector potential we show that the energy stored in a magnetic field is

$$U = \frac{\mu}{2}\int \mathrm{d}^3 r\, \mathrm{d}^3 r_o\, \frac{\boldsymbol{j}(\boldsymbol{r})/c \cdot \boldsymbol{j}(\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{7.8}$$

(e) For a set of current loops $I_a = I_1, I_2, \ldots$ the energy integral is

$$U = \sum_a \frac{1}{2} L_a I_a^2 + \frac{1}{2}\sum_{a\neq b} M_{ab} I_a I_b \tag{7.9}$$

The voltage in the $a-\mathrm{t}h$ loop is

$$\mathcal{E}_a = L_a \frac{dI_a}{dt} + \sum_{b\neq a} M_{ab} \frac{dI_b}{dt} \tag{7.10}$$

(f) For a set of current loops $I_a = I_1, I_2, I_3 \ldots$. Let $\boldsymbol{A}_{\rm ext}$ be the vector potential from all the loops *except* the $a - th$ loop. The energy stored in the $a - th$ loop is from Eq. (7.6) :

$$U_a = \oint I_a \mathrm{d}\boldsymbol{\ell} \cdot \boldsymbol{A}_{\rm ext} = I_a \Phi_B = I_a \int \boldsymbol{B}_{\rm ext} \cdot \mathrm{d}\mathbf{a} \tag{7.11}$$

This is the energy required to increase I_a from zero up to I_a, at *fixed* $\boldsymbol{A}_{\rm ext}$.

7.2 Quasi-static fields: Lecture 20 and 21

(a) We studied a prototypical problem of a charging a capacitor plates. The maxwell equations are categorized by an expansion in $1/c$, *i.e.* that the speed of light is fast compared to L/T the characteristic lengths L and times T. In this approximation the fields are determined instantaneously across space. Organizing the maxwell equations

$$\nabla \cdot \boldsymbol{E} = \rho \tag{7.12}$$

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E} \tag{7.13}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{7.14}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B} \tag{7.15}$$

in powers of $1/c$ we have:

i) 0th order:

$$\nabla \cdot \boldsymbol{E}^{(0)} = \rho \qquad \nabla \times \boldsymbol{B}^{(0)} = 0 \tag{7.16}$$

$$\nabla \times \boldsymbol{E}^{(0)} = 0 \qquad \nabla \cdot \boldsymbol{B}^{(0)} = 0 \tag{7.17}$$

ii) 1st order:

$$\nabla \cdot \boldsymbol{E}^{(1)} = 0 \qquad \nabla \times \boldsymbol{B}^{(1)} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t \boldsymbol{E}^{(0)} \tag{7.18}$$

$$\nabla \times \boldsymbol{E}^{(1)} = 0 \qquad \nabla \cdot \boldsymbol{B}^{(1)} = 0 \tag{7.19}$$

iii) 2nd order:

$$\nabla \cdot \boldsymbol{E}^{(2)} = 0 \qquad \nabla \times \boldsymbol{B}^{(2)} = 0 \tag{7.20}$$

$$\nabla \times \boldsymbol{E}^{(2)} = -\frac{1}{c}\partial_t \boldsymbol{B}^{(1)} \qquad \nabla \cdot \boldsymbol{B}^{(2)} = 0 \tag{7.21}$$

iv) Third order ...

$$\nabla \cdot \boldsymbol{E}^{(3)} = 0 \qquad \nabla \times \boldsymbol{B}^{(3)} = +\frac{1}{c}\partial_t \boldsymbol{E}^{(2)} \tag{7.22}$$

$$\nabla \times \boldsymbol{E}^{(3)} = -\frac{1}{c}\partial_t \boldsymbol{B}^{(2)} \qquad \nabla \cdot \boldsymbol{B}^{(3)} = 0 \tag{7.23}$$

Often time this goes beyond what is needed. Often at 3rd oder and beyond we will need to consider radiation at this order ..., since the fields do not (in general) decay faster than $1/r$ at infinity.

(b) In the quasi-static approximation we find a series of the following form:

$$\boldsymbol{E} = \boldsymbol{E}^{(0)} + \boldsymbol{E}^{(2)} + \ldots \tag{7.24}$$

$$\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(3)} + \ldots \tag{7.25}$$

were $E^{(2)}$ is smaller than $\boldsymbol{E}^{(0)}$ by a factor of $(L/(cT))^2$. Similarly, $\boldsymbol{B}^{(3)}$ is are typically smaller than $\boldsymbol{B}^{(1)}$ (the leading B) by a factor $(L/(cT))^2$. If the material is ferromagnetic then μ can enhance the strength of $\boldsymbol{B}$ relative to the naive estimates.

Quasi-static approximation with gauge-potentials

(a) We often solve for the gauge potentials φ and $\boldsymbol{A}$ (instead of $\boldsymbol{E}$ and $\boldsymbol{B}$) order by order in $1/c$ instead of $\boldsymbol{E}$ and $\boldsymbol{B}$ (see below). For example to second order in the Coulomb gauge we have

i) 0th order:

$$-\nabla^2\varphi = \rho \qquad \text{(actually all orders)} \tag{7.26}$$

ii) 1s order :

$$-\nabla^2\boldsymbol{A} = \frac{\boldsymbol{j}}{c} + \frac{1}{c}\partial_t(-\nabla\varphi) \tag{7.27}$$

This is sufficient to determine the electric and magnetic field to *second order*

$$\boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{A} + \nabla\varphi \tag{7.28}$$

The covariant gauge can be studied similarly:

(b) In the covariant gauge we have

i) 0th:

$$-\nabla^2\varphi^{(0)} = \rho \tag{7.29}$$

ii) 1st:

$$-\nabla^2\boldsymbol{A} = \frac{\boldsymbol{j}}{c} \tag{7.30}$$

Together with gauge constraint:

$$\frac{1}{c}\partial_t\varphi^{(0)} + \nabla\cdot\boldsymbol{A} = 0 \tag{7.31}$$

iii) 2nd:

$$-\nabla^2\varphi^{(2)} = -\frac{1}{c^2}\frac{\partial^2\varphi^{(0)}}{\partial t^2} \tag{7.32}$$

7.3 Quasi-static approximation in metals and skin depth: Lecture 22

(a) For the metals we derived a (quasi-static) diffusion equation for $\boldsymbol{B}$ by taking the curl of Amperes law and using Faraday's law

$$\nabla^2 \boldsymbol{B} = \frac{\sigma\mu}{c^2}\partial_t \boldsymbol{B} \tag{7.33}$$

You should feel comfortable deriving this. This shows the magnetic field diffuses in metal, with diffusion coefficient

$$D = \frac{c^2}{\mu\sigma}\,. \tag{7.34}$$

The diffusion coefficient has units $(distance)^2/time$ and is for copper, $D \sim \frac{1\,cm^2}{millisec}$

(b) Eq. (7.33) should be compared to the diffusion equation for a drop of dye in a cup of water:

$$D\nabla^2 n = \partial_t n\,. \tag{7.35}$$

A Gaussian drop of dye spreads out in time, and the mean squared width of the the drop increases in time as :

$$(\Delta x)^2 = 2D\Delta t \tag{7.36}$$

(c) If the RHS of Eq. (7.33) (the induced current) is small compared to the LHS, then we can neglect the induced currents and the magnetic field is unscreened by the induced currents. In this case, the characteristic lengths L we are considering are shorter than the skin depth:

$$\delta \equiv \sqrt{\frac{2c^2}{\sigma\mu\omega}} \tag{7.37}$$

On length scales larger than δ the magnetic field is damped by induced currents:

$$L \ll \delta \qquad \text{magnetic field unscreened} \tag{7.38}$$

$$L \gg \delta \qquad \text{magnetic field screened} \tag{7.39}$$

At fixed L this can also be expressed in term of frequency, *i.e.* if ω is less than $\omega_{\rm ind} \equiv c^2/\sigma L^2$ then the magnetic field is not screened at length L, but if ω is greater than $\omega_{\rm ind} \equiv c^2/\sigma L^2$, then the magnetic field is screened at length L.