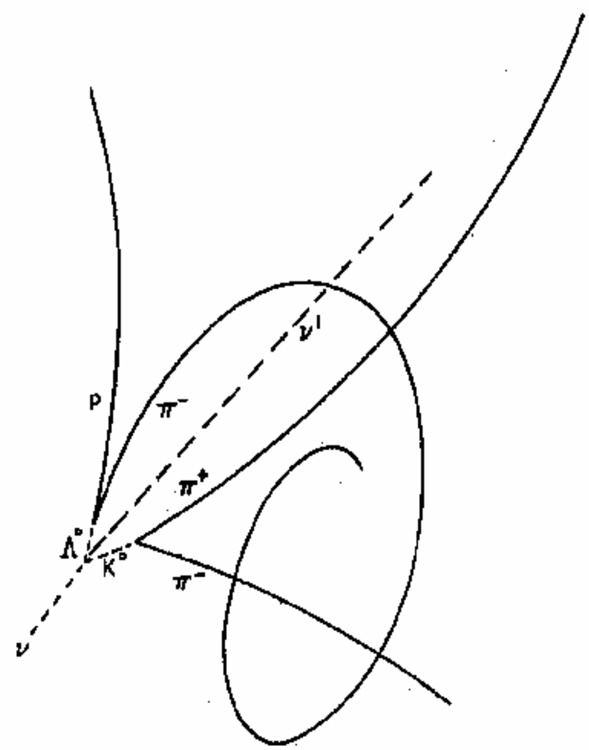
Problem 1. Kinematics of the Lambda decays

The lambda particle (Λ) is a neutral baryon of mass $M = 1115\,\mathrm{MeV}$ that decays with a lifetime of $\tau = 2.9 \times 10^{-10}\,\mathrm{s}$ into a nucleon of mass $m_1 = 939\,\mathrm{MeV}$ and a π-meson of mass $m_2 = 140\,\mathrm{MeV}$. It was first observed by its charged decay mode $\Lambda \to p + \pi^-$ in cloud chambers. In the clould chamber (and in detectors today) the charge tracks seem to appear out of nowhere from a single point (since the lambda is neutral) and have the appearance of the letter vee. Hence this decay is known as a vee decay. The particles' identities and momenta can be inferred from their ranges and curvature in the magnetic field of the chamber. (In this problem M, m_1, m_2 etc are short for Mc^2, m_1c^2, m_2c^2 etc., and p_1 and p_2 are short for cp_1 and cp_2) A picture of the vee decay is shown below



(a) Using conservation of momentum and energy and the invariance of scalar products of four vectors show that, if the opening angle θ between the two tracks is measured, the mass of the decaying particle can be found from the formula

$$M^2 = m_1^2 + m_2^2 + 2\,(E_1E_2 - p_1p_2\cos\theta)$$

(b) A lambda particle is created with total energy of 10 GeV in and moves along the x-axis. How far on the average will it travel in the chamber before decaying? (Answer: 0.78 m)

(c) Show that the momentum of the pion (or the proton) in the rest frame of the Lambda is

$$p_1 = p_2 = \sqrt{\frac{(M^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}{4M^2}} \tag{1}$$

and evaluate the velocity/c of the pion v_π/c numerically. (Answer: 0.573)

Use this to determine if a pion emitted in the negative x direction in the frame of the decaying 10 GeV lambda will move forward (positive-x) or backwards (negative-x) in the lab frame.

(d) What range of opening angles will occur for a 10 GeV lambda if the decay is more or less isotropic in the lambda's rest frame? (Answer $\theta = 0 \ldots 4.32^o$)

Problem 2. Proper acceleration

A particle of mass m, starting at rest at time $t = 0$ and $x = 0$ in the lab frame, experiences a constant acceleration, a, in the x-direction in its own rest frame.

(a) The acceleration four vector

$$A^\mu \equiv \frac{d^2 x^\mu}{d\tau^2} \tag{2}$$

is specified by the problem statement. What are the four components of the acceleration four vector in the rest frame of the particle and in the lab frame. What is the acceleration, d^2x/dt^2, in the lab frame.

(b) Show that the position of the particle as a function of time can be parameterized by a real number p

$$x = \frac{c^2}{a}\left[\cosh(p) - 1\right] \tag{3}$$

where p is related to the time t through the equation:

$$c\,t = \frac{c^2}{a}\sinh(p) \tag{4}$$

(c) Show that the parameter p is proportional proper time, $p = \frac{a}{c}\tau$.

(d) The rapidity of a particle, Y, is defined by its velocity

$$\frac{v}{c} \equiv \tanh(Y) \tag{5}$$

where $v = dx/dt$. Show that the four velocity $u^\mu = dx^\mu/d\tau$ is related to the rapidity through the hyperbolic relations.

$$(u^0/c, u^1/c) = (\cosh(Y), \sinh(Y)) \tag{6}$$

(e) Show that $Y = a\tau/c$

Remark: We see that the rapidity of the particle increases linearly with proper time during proper acceleration.

(f) If the particle has a constant decay rate in its own frame of Γ, show that the probability that the particle survives at late time t is approximately

$$\left(\frac{2at}{c}\right)^{-\Gamma c/a}$$

Problem 3. The stress tensor from the equations of motion

In class we wrote down energy and momentum conservation in the form

$$\frac{\partial \Theta^{\mu\nu}_{\rm mech}}{\partial x^\mu} = F^{\nu}_{\ \rho}\frac{J^\rho}{c} \tag{7}$$

Where the $\nu = 0$ component of this equation reflects the work done by the E&M field on the mechanical constituents, and the spatial components ($\nu = 1, 2, 3$) of this equation reflect the force by the E&M field on the mechanical constituents.

(a) Verify that

$$F^{\nu}_{\ \rho}\frac{J^\rho}{c} = \begin{cases} \boldsymbol{J}/c\cdot\boldsymbol{E} & \nu = 0 \\ \rho E^j + (\boldsymbol{J}/c\times\boldsymbol{B})^j & \nu = j \end{cases} \tag{8}$$

(b) (**Optional**) Working within the limitations of magnetostatics where

$$\nabla\times\boldsymbol{B} = \frac{\boldsymbol{J}}{c} \qquad \nabla\cdot\boldsymbol{B} = 0 \tag{9}$$

show that the magnetic force can be written as the divergence of the magnetic stress tensor, $T^{ij}_B = -B^iB^j + \frac{1}{2}\delta^{ij}B^2$:

$$(\frac{\boldsymbol{J}}{c}\times\boldsymbol{B})^j = -\partial_i T^{ij}_B \tag{10}$$

(c) Consider a solenoid carrying current I with n turns per length, what is the force per area on the sides of the solenoid.

Some reasonable people (on the exam) tried to compute this by computing B and then computing the Lorentz force on the wire. This gives twice the correct answer obtained using the stress tensor. Explain why.

(d) Using the equations of motion in covariant form

$$-\partial_\mu F^{\mu\rho} = \frac{J^\rho}{c} \tag{11}$$

and the Bianchi Identity

$$\partial_\mu F_{\sigma\rho} + \partial_\sigma F_{\rho\mu} + \partial_\rho F_{\mu\sigma} = 0 \tag{12}$$

show that

$$F^{\nu}_{\ \rho}\frac{J^\rho}{c} = -\frac{\partial}{\partial x^\mu}\Theta^{\mu\nu}_{\rm em} \tag{13}$$

where

$$\Theta^{\mu\nu}_{\rm em} = F^{\mu\rho}F^{\nu}_{\ \rho} + g^{\mu\nu}\left(-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{14}$$

Hint: use the fact that $F^{\mu\rho}$ is anti-symmetric under interchange of μ and ρ.

(e) (**Optional**) Verify by direct substitution, using $F^{ij} = \epsilon^{ijk}B_k$, that if there is no electric field that

$$\Theta^{ij} = T^{ij}_B\,. \tag{15}$$

Problem 4. An alternative lagrangian for the point particle:

The action of the point particle is taken to be

$$S_o + S_{\rm int} = \frac{1}{2}\int dp \left[\frac{1}{\eta(p)} mc\, \dot{X}^\mu \dot{X}_\mu - mc\,\eta(p)\right] + \frac{e}{c}\int dp\, \dot{X}^\mu A_\mu(X(p))\,. \tag{16}$$

Here $X^\mu(p)$ is the path of the particle, which is also known as the world-line. $X^\mu(p)$ is parameterized by the real parameter p, and $\dot{X}^\mu = dX^\mu/dp$ is the derivative of the path.

- $\eta(p)$ is known as the tetrad. The tetrad is an additional world line (metric) field that each particle carries around, in much the same way that the particle carries around its spin. The tetrad measures how an infinitessimal change in the world-line parameter dp, is related to an infinitessimal change in the physical propper time, $d\tau$

  $$c\, d\tau = \eta(p) dp \tag{17}$$

  Heuristically we have:

  $$c\, d\tau = \eta(p) dp \qquad c\, d\tau = \frac{(cd\tau)^2}{(cd\tau)} = \frac{-(dp)^2\, \dot{X}^\mu \dot{X}_\mu}{\eta(p) dp} \tag{18}$$

  Thus the action S_o is recognizable as the action of point particle that we had in class:

  $$S_o = \frac{1}{2}\int \underbrace{\frac{dp}{\eta(p)} mc\, \dot{X}^\mu \dot{X}_\mu}_{-mc^2 d\tau} + \underbrace{-mc\,\eta(p) dp}_{-mc^2 d\tau} \tag{19}$$

  $$= -\int mc^2 d\tau \tag{20}$$

- Under reparametrization of the path $p(\bar{p})$, the tetrad changes, so that physical propper time remains invariant:

  $$\bar{\eta}(\bar{p}) \equiv \eta(p) \frac{dp}{d\bar{p}}\,. \tag{21}$$

  i.e.

  $$cd\tau = \eta(p) dp = \bar{\eta}(\bar{p}) d\bar{p} = cd\tau \tag{22}$$

- The equations of motion for the point particle and its tetrad (which is a dynamical field) are found by varying the action,

  $$X^\mu(p) \to X^\mu(p) + \delta X^\mu(p) \tag{23}$$

  $$\eta(p) \to \eta(p) + \delta\eta(p) \tag{24}$$

This problem will study this Lagrangian.

(a) Show that the action

$$S_o + S_{\rm int} \tag{25}$$

is reparametrization invariant, *i.e.* that the value of the action integral with dp, $X(p)$, $\eta(p)$ equals the action integral with $d\bar{p}$, $X(\bar{p})$, $\bar{\eta}(\bar{p})$.

(b) By varying the action show that we obtain the correct equations of motion:

$$m\frac{d^2X^\mu}{d\tau^2} = \frac{e}{c}F^\mu_{\ \nu}\frac{dX^\nu}{d\tau} \tag{26}$$

and the appropriate constraint:

$$\eta(p) = \sqrt{-\dot{X}^\mu \dot{X}_\mu} \equiv c\frac{d\tau}{dp} \tag{27}$$

Problem 5. Fields from moving particle

The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$ were given in class. Choose the axes so that the charge moves along the $z-$axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define a transverse vector (or impact parameter) $\boldsymbol{b}_\perp = (x, y)$, with components x and y. Consider the fields and the source in the limit $\beta \to 1$

(a) As the charge q passes by a charge e at impact parameter $\boldsymbol{b}$, show that the accumulated transverse momentum transfer (transverse impulse) to the charge e during the passage of q is

$$\Delta \boldsymbol{p}_\perp = \frac{eq}{2\pi} \frac{\boldsymbol{b}_\perp}{b_\perp^2 c} \tag{28}$$

(b) Show that the time integral of the absolute value of the longitudinal force to a charge e at rest at an impact parameter $b_\perp$ is

$$\frac{eq}{2\pi\gamma b_\perp c} \tag{29}$$

and hence approaches zero as $\beta \to 1$.

(c) Show that the fields of charge q can be written for $\beta \to 1$ as

$$\boldsymbol{E} = \frac{q}{2\pi} \frac{\boldsymbol{b}_\perp}{b_\perp^2} \delta(ct - z) \,, \qquad \boldsymbol{B} = \frac{q}{2\pi} \frac{\hat{\mathbf{v}}/c \times \boldsymbol{b}_\perp}{b_\perp^2} \delta(ct - z) \,. \tag{30}$$

(d) Show by explicit substitution into the Maxwell equations that these fields are consistent with the 4$-$vector source density

$$J^\alpha = q v^\alpha \delta^2(\boldsymbol{b}_\perp) \delta(ct - z) \tag{31}$$

where $v^\alpha = (c, \hat{\mathbf{v}})$.

Problem 6. (Optional, but good) Kinematics of a Relativistic Rod

Consider a rod of rest length D_o. According to an inertial frame K' the rod is aligned along the x'-axis, and moves moves with velocity u' along the y' axis. The frame K' is moving to the right with velocity v relative to K in the x direction. The coordinate origins of the K and K' systems are chose so that the midpoint of the rod crosses the spatial origin at time $t = t' = 0$, *i.e.* that space-time location of the rod center intersects $t = t' = x = x' = y = y' = 0$.

(a) Find the space-time trajectory of the endpoints of the rod in frame K.

(b) At $t = 0$ in frame K, Show that the angle of the rod to the x-axis is

$$\phi = -\mathrm{atan}(\gamma_v v u'/c^2) \tag{32}$$

where $\gamma_v = 1/\sqrt{1-(v/c)^2}$

(c) Show that the length of the rod in frame K is

$$\sqrt{\left(\frac{D_o}{\gamma}\right)^2 + \left(\frac{vu'}{c^2}\right)^2 D_o^2}$$