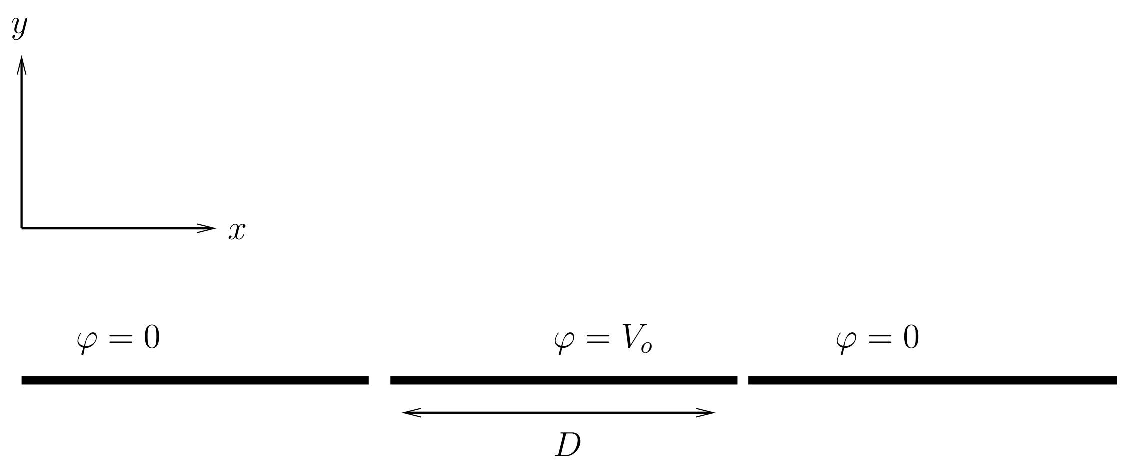


Problem 1. Potential from a strip.

An infinite conducting strip of width D (between $0 < x < D$) is maintained at potential V_o, while on either side of the strip are grounded conducting planes. The strip and the planes are separated by a tiny gap as shown below.

(a) Following a similar example given in class, determine the potential everywhere in the upper half plane $y > 0$.

(b) Determine the surface charge density on the strip and on the grounded planes, and make a graph.

Problem 2. A Periodic Array of Charged Rings.

Consider a periodic array of charged rings of radius R and separation b, so that the z-coordinates of the rings are $z = 0, \pm b, \pm 2b, \ldots$. Each ring has charge Q. We will find the potential below

(a) This problem is solved by exploiting the periodic nature of the problem, writing the charge density and the potential as a fourier series. Use completeness to show that that the charge density is

$$\rho(\boldsymbol{x}) = \frac{Q}{2\pi R}\delta(\rho - R)\frac{1}{b}\sum_{n=-\infty}^{\infty} e^{ik_n z} \tag{1}$$

where $k_n = 2\pi n/b$.

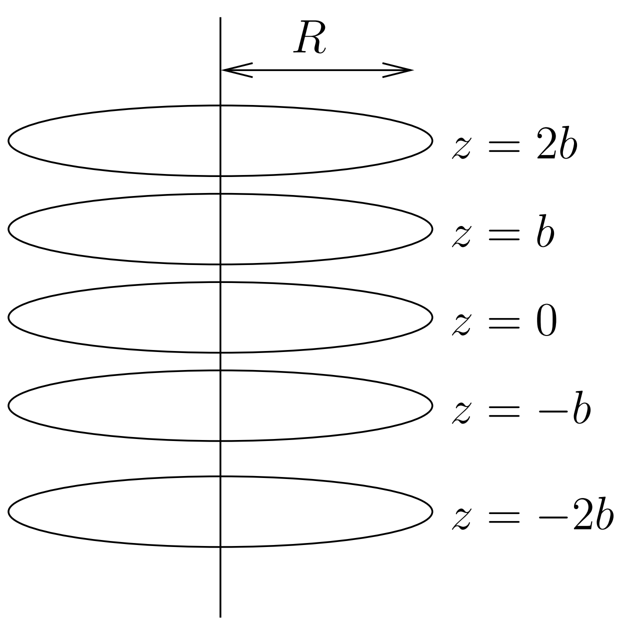
(b) Solve for the potential inside and outside the rings, and use the jump condition to relate the two solutions. Show that the potential outside of the rings is

$$\varphi(\boldsymbol{x}) = \frac{Q}{2\pi b}\left[-\ln\rho + 2\sum_{n=1}^{\infty}\cos(k_n z) I_0(k_n R) K_0(k_n \rho)\right] \tag{2}$$

(c) For ρ large show that

$$\varphi(\boldsymbol{x}) \simeq \frac{Q}{2\pi b}\left[-\ln\rho + \sqrt{\frac{b}{\rho}}\cos(\frac{2\pi z}{b}) I_o(2\pi R/b) e^{-2\pi\rho/b}\right] \tag{3}$$

and explicitly interpret the leading term, $-\ln\rho$, and its coefficient, $Q/(2\pi b)$.



Problem 3. A point charge and a semi-infinite dielectric slab

A point charge of charge q in vacuum is at the origin $\boldsymbol{r}_o = (0, 0, 0)$. It is separated from a semi-infinite dielectric slab filling the space $z > a$ with dielectric constant $\epsilon > 1$. When evaluating the potential for $z < a$, an image charge solution is found by placing an image charge at $z = 2a$. When evaluating the potential for $z > a$ we place an image charge at the origin. The full image solution is

$$\varphi(\boldsymbol{r}) = \begin{cases} \frac{q}{4\pi|\boldsymbol{r}|} - \frac{\beta q}{4\pi|\boldsymbol{r}-2a\hat{\boldsymbol{z}}|} & z < a \\ \frac{\beta' q}{4\pi\epsilon|\boldsymbol{r}|} & z > a \end{cases} \tag{4}$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$ and $\beta' = (2\epsilon)/(1 + \epsilon)$

(a) Sketch a picture of the resulting electric field lines.

(b) Quite generally show that the electric field lines refract at a discontinuous interface

$$\frac{\tan\theta_I}{\epsilon_{\rm I}} = \frac{\tan\theta_{\rm II}}{\epsilon_{\rm II}} \tag{5}$$

where $\theta_{\rm I}$ and $\theta_{\rm II}$ are the angles between the normal pointing from I to II and the electric fields in region I and region II, and $\epsilon_{\rm I}$ and $\epsilon_{\rm II}$ are the dielectric constants.

Problem 4. A Dielectric slab intervenes.

This problem will calculate the force between a point charge q in vacuum and a dielectric slab with dielectric constant $\epsilon > 1$. The point charge is at the origin $\boldsymbol{r}_o = (x_o, y_o, z_o) = (0,0,0)$, but we will keep x_o, y_o, z_o for clarity. The slab lies between $z = a$ and $z = a + \delta$ with $a > 0$ and has infinite extent in the x, y directions

(a) Write the free space Green function as a Fourier transform

$$\frac{q}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = q \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp \cdot (\boldsymbol{r}_\perp - \boldsymbol{r}_{o\perp})} g^o_{\boldsymbol{k}_\perp}(z, z_o) \tag{6}$$

and show that the free space green function in fourier space is

$$g^o_{\boldsymbol{k}_\perp}(z, z_o) = \frac{e^{-k_\perp|z - z_o|}}{2k_\perp} \tag{7}$$

(b) Now consider the dielectric slab and write the potential produced by the point charge at $z_o = 0$ as a Fourier transform

$$\varphi(\boldsymbol{r}_\perp, z) = q \int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp \boldsymbol{r}_\perp} g_{\boldsymbol{k}_\perp}(z)\,, \tag{8}$$

and determine for $g_{k_\perp}(z)$ by solving in each region, matching across the interfaces, and by analyzing the jump at z_o. Show that for $z < 0$ and $0 < z < a$

$$g_{\boldsymbol{k}_\perp}(z) = \begin{cases} \frac{e^{kz}}{2k} - \frac{\beta e^{k(z-2a)}(1 - e^{-2\delta k})}{2k(1 - \beta^2 e^{-2\delta k})} & z < 0 \\ \frac{e^{-kz}}{2k} - \frac{\beta e^{k(z-2a)}(1 - e^{-2\delta k})}{2k(1 - \beta^2 e^{-2\delta k})} & 0 < z < a \end{cases} \tag{9}$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$ and we have written $k = k_\perp$ to lighten the notation.

(c) Checks:

(i) Show that for $\delta \to \infty$ the potential for $z < a$ is in agreement with the results of the previous problem.

(ii) Show that when $\epsilon \to \infty$ (when the dielectric becomes almost metallic) you get the right potential.

(d) Show that the electric potential for region $z < a$ can be written

$$\varphi = \varphi_{\text{ind}} + \frac{q}{4\pi r} \tag{10}$$

where φ_{ind} is the induced potential and is regular at $r = 0$. Show that the force on the point charge is

$$F^z = \beta \frac{q^2}{4\pi(2a)^2} \int_0^\infty du\, \frac{4ue^{-2u}(1 - e^{-2(\delta/a)u})}{1 - \beta^2 e^{-2(\delta/a)u}} \tag{11}$$

(e) Use a program such as mathematica to make a graph of the force $F^z/(\beta q^2/(4\pi(2a)^2))$ versus δ/a for $\beta = 0.1, 0.5, 0.9$ and sketch the result.