Problem 1. A cylinder in a magnetic field

A very long hollow cylinder of inner radius \( a \) and outer radius \( b \) of permeability \( \mu \) is placed in an initially uniform magnetic field \( \mathbf{B}_o \) at right angles to the field.

(a) For a constant field \( \mathbf{B}_o \) in the \( x \) direction show that \( \mathbf{A} = \mathbf{B}_o y \) is the vector potential. This should give you an idea of a convenient set of coordinates to use.

**Remark:** See Wikipedia for a list of the vector Laplacian in all coordinates. Most often the vector Laplacian is used if the current is azimuthal and solutions may be looked for with \( A_\phi \neq 0 \) and \( A_r = A_\theta = 0 \) (or \( A_\rho = A_z = 0 \) in cylindrical coordinates). This could be used for example in Problem 3. Similarly if the current runs up and down, with \( A_z \neq 0 \) and \( A_\rho = A_\phi = 0 \), so that \( \mathbf{B} = (B_x(x, y, z), B_y(x, y, z), 0) \) is independent of \( z \), then the vector Laplacian in cylindrical coordinates \(-\nabla^2 A_z\) is a good way to go.

(b) Show that the magnetic field in the cylinder is constant \( \rho < a \) and determine its magnitude.

(c) Sketch \( |\mathbf{B}|/|\mathbf{B}_o| \) at the center of the as function of \( \mu \) for \( a^2/b^2 = 0.9, 0.5, 0.1 \) for \( \mu > 1 \).
Problem 2. Helmholtz coils

Consider a compact circular coil of radius \(a\) carrying current \(I\), which lies in the \(x-y\) plane with its center at the origin.

(a) By elementary means compute the magnetic field along the \(z\) axis.

(b) Show by direct analysis of the Maxwell equations \(\nabla \cdot \mathbf{B} = 0\) and \(\nabla \times \mathbf{B} = 0\) that slightly off axis near \(z = 0\) the magnetic field takes the form

\[
B_z \simeq \sigma_0 + \sigma_2 \left( z^2 - \frac{1}{2} \rho^2 \right), \quad B_\rho \simeq -\sigma_2 z \rho, \tag{1}
\]

where \(\sigma_0 = (B^0_z)\) and \(\sigma_2 = \frac{1}{2} \left( \frac{\partial^2 B^0_z}{\partial z^2} \right)\) are the field and its \(z\) derivatives evaluated at the origin. For later use give \(\sigma_0\) and \(\sigma_2\) explicitly in terms of the current and the radius of the loop.

**Remark:** Upon solving this problem, it should be clear that this method of solution does not rely on being close to \(z = 0\). We just chose \(z = 0\) for definiteness.

(c) Now consider a second identical coil (co-axial with the first), having the same magnitude and direction of the current, at a height \(b\) above the first coil, where \(a\) is the radii of the rings. With the coordinate origin relocated at the point midway between the two centers of the coils, determine the magnetic field on the \(z\)-axis near the origin as an expansion in powers of \(z\) to \(z^4\). Use mathematica if you like. You should find that the coefficient of \(z^2\) vanishes when \(b = a\).

**Remark** For \(b = a\) the coils are known as Helmholtz coils. For this choice of \(b\) the \(z^2\) terms in part (c) are absent. (Also if the off-axis fields are computed along the lines of part (b), they also vanish.) The field near the origin is then constant to 0.1\% for \(z < 0.17 a\).
Problem 3. The field from a ring current.

Consider conducting ring of current radius \( a \) lying in the \( x - y \) plane, carrying current \( I \) in the counter clockwise direction, \( \mathbf{I} = I \hat{\phi} \).

(a) Starting from the general (coulomb gauge) expression

\[
\mathbf{A}(\mathbf{r}) = \int d^3 \mathbf{r}_o \frac{\mathbf{j}(\mathbf{r}_o)/c}{4\pi|\mathbf{r} - \mathbf{r}_o|} 
\]

and the expansion of \( 1/(4\pi|\mathbf{r} - \mathbf{r}_o|) \) in spherical coordinates, show that the expansion of \( A_\phi \) in the \( x, y \) plane inside the ring is

\[
A_\phi(\rho)|_{z=0} = \frac{I}{2c} \sum_{\ell=1}^{\infty} \frac{(P^1_\ell(0))^2}{\ell(\ell+1)} \left( \frac{\rho}{a} \right)^\ell 
\]

where \( \rho = \sqrt{x^2 + y^2} \) and \( P^1_\ell \) is the associated Legendre polynomial. (Check out wikipedia entry on spherical harmonics)

(b) Compute \( B_z(\rho) \) in the \( x, y \) plane.

(c) Show that close to the axis of the shell the magnetic field you computed in part (b) is in agreement with the results of Eq. (1) when evaluated at \( z = 0 \), i.e. that for small \( \rho \) part (b) yields \( B_z(\rho) \approx \sigma_0 - \frac{1}{2}\sigma_2 \rho^2 \) with the appropriate values of \( \sigma_0 \) and \( \sigma_2 \).

Remark: Using the generating function of Legendre polynomials derived in class

\[
\frac{1}{\sqrt{1 + r^2 - 2r \cos \theta}} = \sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos \theta) 
\]

and the definition of \( P^1_\ell(\cos \theta) = -\sin \theta \frac{dP_\ell(\cos \theta)}{d(\cos \theta)} \), we show that

\[
\sum_{\ell=1}^{\infty} r^\ell P^1_\ell(0) = \frac{-r}{(1 + r^2)^{3/2}} \approx -r + \frac{3}{2} r^3 - \frac{15}{8} r^5 + \ldots 
\]

establishing that

\[
P^1_1(0) = -1 \quad P^1_3(0) = \frac{3}{2} \quad P^1_5(0) = -\frac{15}{8} \quad P^1_\ell(0) = 0 \text{ for } \ell \text{ even}.
\]

(d) Consider a magnetic dipole of magnetic moment \( \mathbf{m} = -m \hat{z} \) in the \( x - y \) plane oriented oppositely to the field from the ring, show that when the dipole is inside the ring the force on the dipole is

\[
\mathbf{F} = -\mathbf{\rho} \frac{mB_o}{a} \sum_{\ell=3}^{\infty} \frac{(\ell - 1)}{\ell} (P^1_\ell(0))^2 \left( \frac{\rho}{a} \right)^{\ell-2} 
\]

where the negative indicates that the force is towards the center, and \( B_o = I/(2ca) \) is the magnetic field in the center of the ring.

(e) Plot the force \( |\mathbf{F}|/[mB_o/a] \) as a function of \( \rho/a \).
Problem 4. Two electrodes in a conductor filling half of space

Two small spherical electrodes of radius $a$ are embedded in a semi-infinite medium of conductivity $\sigma$, each at a distance $d > a$ from the plane face of the medium and at a distance $b$ from each other. Find the resistance between the electrodes. Sketch the flow lines of current if the two electrodes are held at a potential difference $\Delta \varphi$. 
Problem 5. Force on a displaced sphere

A hollow metal spherical shell of radius, $a$, raised to potential $V_o$ (relative to zero at infinity) is placed inside a spherical cavity of radius $b$ (with $b > a$), which is carved out of an infinite block of dielectric of dielectric constant $\epsilon$. The metal sphere is displaced from the center of the cavity by a small distance $s = s \hat{z}$.

(a) Determine the potential to zeroth order in $s$ (see the solutions to first order given below)

(b) Show that to first order in $s$ the potential outside the shell can be written:

$$\varphi_{\text{in}} = \varphi_o + \frac{Q}{4\pi} \left[ \frac{1}{r} + \left( \frac{a^3}{b^3\beta - a^3} \right) \left( \frac{s}{r^2} - \frac{sr}{a^3} \right) \cos \theta \right] \quad r < r_s(\theta) \quad (8)$$

$$\varphi_{\text{out}} = \frac{Q}{4\pi\epsilon} \left[ \frac{1}{r} + \frac{s}{r^2} \left( \frac{a^3 + \frac{1}{2}b^3}{b^3\beta - a^3} \right) \cos \theta \right] \quad r > r_s(\theta) \quad (9)$$

Here we have defined two reappearing constants

$$\beta \equiv \frac{(1 + 2\epsilon)/(\epsilon - 1)}{\epsilon - 1} \quad (10)$$

$$\varphi_o \equiv -\frac{Q}{4\pi b} \frac{\epsilon - 1}{\epsilon} \quad (11)$$

Finally $Q$ is the induced charge on the surface of the (inner) sphere which is related to the potential $V_o$ by $Q = 4\pi V_o ab\epsilon/(b\epsilon - a(\epsilon - 1))$.

(Hint: take the center of coordinates to be the center of the metal spherical shell. Show that the dielectric boundary is at

$$r_s(\theta) \approx b - s \cos \theta + O(s^2) \quad (12)$$

Then show that two unit vectors parallel and perpendicular to the surface are (respectively)

$$\mathbf{u} \approx \hat{\theta} + \frac{s}{b} \sin \theta \hat{r} + O(s^2) \quad (13)$$

$$\mathbf{n} \approx -\frac{s}{b} \sin \theta \hat{\theta} + \hat{r} + O(s^2) \quad (14)$$

Use these vectors to write down the boundary conditions through first order in $s$ at $r = r_s(\theta)$. Then solve for the fields in a power series in $s$, adjusting the coefficients to satisfy the boundary conditions order by order.

(c) Show that the force on the shell to first order in $s$ is

$$F^z = \frac{Q^2}{4\pi b^3\beta - a^3} \frac{s}{b^3\beta - a^3} \quad (15)$$