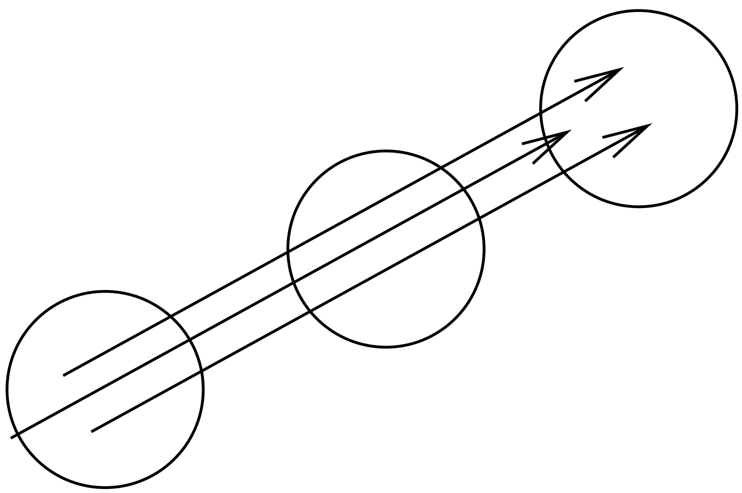
Problem 1. Azimuthal eddy currents in a wire

A longitudinal AC magnetic field $\boldsymbol{B}(t) = \hat{\mathbf{z}} B_o \cos(\omega t)$ is driven through the interior of a (solid) ohmic tube with length L and radius $R \ll L$, drawn rather schematically below.

(a) Find the low frequency eddy-current density inside the tube, neglecting the effects of self inductance.

(b) Find the correction to the eddy-current density produced by self-inductance (to next order in ω).

(c) Derive the condition where the correction in (b) can be ignored.



Problem 2. A rotating magnet

The equation of motion for a magnetic dipole moment $\boldsymbol{m}$ which rotates about its center with angular velocity $\boldsymbol{\Omega}$ is $d\boldsymbol{m}/dt = \boldsymbol{\Omega} \times \boldsymbol{m}$.

(a) Find the electric and magnetic fields associated witht this object to lowest nontrivial order in inverse powers of c.

(b) Give a parametric estimate the magnitude of E/B at a given radius r (i.e. something of the form $E/B \sim$ (products of dimensionful parameters)) . At what radius is the solution you found in part (a) valid? At what radius does it break down and why?

Problem 3. Electric and Magnetic fields of AC Solenoid

A cylindrical solenoid of high conductivity and radius a carries surface current $\mathbf{K} = K_o \cos(\omega t)\hat{\boldsymbol{\phi}}$

(a) Determine the electric and magnetic fields to the first non-vanishing order in the quasi-static approximation.

(b) Show that the magnetic field to the next-to-leading order in the quasi-static approximation outside the cylinder is

$$\Delta B = \delta B_z(\rho) - \delta B_z(\rho_{\max}) = \frac{K_o}{c} \cos(\omega t)\ \tfrac{1}{2}(\omega a/c)^2 \left(-\log\frac{\rho}{a} + C\right) \tag{1}$$

where C $= \log \rho_{\max}/a$. Here we are quoting ΔB the difference between δB at ρ and δB at $\rho_{\max}$.

(c) The cutoff $\rho_{\max}$ arises because the quasi static approximation breaks down for large ρ where the physics of radiation becomes important. $\rho_{\max}$ should be of order $\rho_{\max} \sim c/\omega$. Explain qualitatively why the approximation breaks down for this radius.

Remark: Certainly $|\delta B_z(\rho_{\max})|$ is logarithmically smaller than $|\delta B_z(\rho)|$ for $\rho \sim a$. In a logarithmic approximation we can neglect $\delta B_z(\rho_{\max})$ and set $\rho_{\max} = c/\omega$ leading to

$$\delta B_z(\rho) \simeq \frac{K_o}{c} \cos(\omega t)\ \tfrac{1}{2}(\omega a/c)^2 \left(-\log\frac{\rho}{a} + \log(c/(\omega a))\right) \tag{2}$$

Leading log accuracy may not be familiar to you. It just says that we are neglecting the constant inside the logarithm which is of order 1. Thus in this approximation,

$$\log(100/2) = \log(100) - \log(2) \simeq \log(100) \tag{3}$$

$$3.9 \simeq 4.6 \tag{4}$$

which is often good enough for government work. Bethe famously used such approximations to estimate the first QED corrections to the hydrodgen spectrum.

(d) Determine the magnetic field (to the next-to-leading order in the quasi-static approximation) inside the cylinder to logarithmic accuracy, and qualitatively sketch the complete magnetic field $B(\rho)/B_o$ where B_o is the leading order answer in the center of the cylinder.

Remark: Note that the ρ depence of part (b) and part (c) does not depend on the value of $C = \log(\rho_{\max}/a)$.

(e) Determine the vector and scalar potentials in the Coulomb and Lorentz gauges to the required order and accuracy to reproduce the electric and magnetic fields in part (a) and verify that you obtain the correct fields.

Problem 4. (due friday) Eddy-Current Levitation

A wire loop of radius b in the $x - y$ plane carries a time-harmonic current $I_o \cos \omega t$. Find the value of I_o needed to levitate a small sphere of mass m, radius a, and conductivity σ at a height z above the center of the loop. Assume $a \ll b$ and that $\delta \ll a$ where δ is the skin depth of the sphere.

Problem 5. (optional fun problem) Dipole down the tube

A small magnet (weight w) falls under gravity down the center of an infinitely long, vertical, conducting tube of radius a, wall thickness $D \ll a$, and conductivity σ. Let the tube be concentric with the $z-$axis and model the magnet as a pointlike dipole with moment $\boldsymbol{m} = m\hat{\mathbf{z}}$. We can find the terminal velocity of the magnet by balancing its weight agains the magnetic drag force associated with the ohmic loss in the walls of the tube.

(a) At the moment it passes through $z = z_o$, show that the magnetic flux produced by $\boldsymbol{m}$ through the a ring of radius a at height z' is

$$\Phi_B = \frac{m}{2}\frac{a^2}{r_o^3} \qquad \text{where} \qquad r_o^2 = a^2 + (z_o^2 - z')^2 \tag{5}$$

(b) When the speed v of the dipole is small, argue that the Faraday EMF induced in the ring is

$$\mathcal{E} = -\frac{1}{c}\partial_t \Phi_B = \frac{v}{c}\frac{\partial \Phi_B}{\partial z'} \tag{6}$$

(c) Show that the current induced in the thin slice of tube which includes the ring is

$$dI = \frac{3mav\sigma D}{4\pi}\frac{(z_o - z')}{r_o^5}dz' \tag{7}$$

(d) Compute the magnetic drag force $\mathbf{F}$ on $\boldsymbol{m}$ by equating the rate at which the force does work to the power dissipated in the walls of the tube by Joule heating.

(e) Find the terminal velocity of the magnet.