Problem 1. Green theorem for first and second order equations and the initial value problem

First order: Consider a model first order equation equation for the velocity

$$m\frac{dv}{dt} + m\eta v = 0 \tag{1}$$

describing how a particle slows down.

(a) Determine the Green function for this equation, *i.e.* find the causal function that satisfies

$$\left[m\frac{d}{dt} + m\eta\right] G_R(t) = \delta(t) \tag{2}$$

using the direct method, and by fourier transforms.

(b) Show that the solution at time t satisfying the boundary conditions specified at $t = t_o$ are

$$v(t) = mG_R(t, t_o)v(t_o) \tag{3}$$

This is normally how the Green function (propagator) is used in quantum mechanics. The Green function is used slightly differently for second order equations, since x and $\dot{x}$ enter the game.

Second order: In class we showed that the electric potential can be determined from knowledge of the boundary value and the Green function. A very similar statement can be made about an initial value problem, *i.e.* the solution at future times can be determined from the initial conditions and the Green function.

For definiteness we will take a harmonic oscillator with mass m and resonant frequency ω_o:

$$m\frac{d^2x}{dt^2} + m\omega_o^2 x = 0$$

The retarded Green function $G(t|t_o)$ is the position $x(t)$ of the harmonic oscillator at time t from an impulsive force at time t_o. It is causal, meaning that it vanishes whenever $t < t_o$, *i.e.*

$$\left(m\frac{d^2}{dt^2} + m\omega_o^2\right) G_R(t|t_o) = \delta(t - t_o) \qquad \text{and } G_R(t, t_o) = 0 \text{ for } t < t_o \tag{4}$$

(a) Given the initial conditions for the oscillator, $x(t_o)$ and $\partial_{t_o} x(t_o)$, at time t_o show that the future value of the oscillator $x(t)$ is given by the Wronskian of the Green function and the initial conditions

$$x(t) = m\left[G_R(t, t_o)\partial_{t_o} x_o - x(t_o)\partial_{t_o} G_R(t, t_o)\right] \qquad t > t_o \tag{5}$$

Hint use the EOM to prove Greens theorem, *i.e.* that the wronskian of the Green function and the solution we are looking for satisfies

$$\partial_{t_o}\left[x(t_o)\left(m\partial_{t_o} G_R(t, t_o)\right) - \left(m\partial_{t_o} x(t_o)\right) G_R(t, t_o)\right] = x(t_o)\delta(t - t_o)\,. \tag{6}$$

Then use this result together with the fact that G_R satisfies retarded boundary conditions to prove Eq. (5) . We also tacitly assume that $G_R(t, t_o)$ satisfies

$$\left(m\frac{d^2}{dt_o^2} + m\omega_o^2\right) G_R(t|t_o) = \delta(t - t_o) \qquad \text{and } G_R(t, t_o) = 0 \text{ for } t < t_o \tag{7}$$

which is true because the harmonic oscillator is self adjoint.

You could also proceed directly, showing that Eq. (5) satisfies the equations of motion

$$\left(m\frac{d^2}{dt^2} + m\omega_o^2\right) x(t) = 0 \tag{8}$$

and the initial conditions,

$$\lim_{t\to t_o} x(t) = x(t_o) \tag{9}$$

$$\lim_{t\to t_o} \frac{dx(t)}{dt} = \partial_{t_o} x(t_o) \tag{10}$$

(b) Use the Green function for the undamped oscillator given in class to verify that you get the correct result for $x(t)$ in terms of the initial conditions.

(c) Show that for the wave equation, $-\Box G_R(t\boldsymbol{x}|t_o\boldsymbol{x}_o) = \delta(t-t_o)\delta^3(\boldsymbol{x}-\boldsymbol{x}_o)$, the appropriate generalization is

$$u(t, \boldsymbol{x}) = \frac{1}{c^2}\int d^3\boldsymbol{x}_o \left[G(t\boldsymbol{x}|t_o\boldsymbol{x}_o)\partial_{t_o} u(t_o, \boldsymbol{x}_o) - u(t_o, \boldsymbol{x}_o)\partial_{t_o} G(t\boldsymbol{x}|t_o\boldsymbol{x}_o)\right] \tag{11}$$

Remark: The results of this problem show that the general solution to the driven damped harmonic oscillator starting from some initial time moment t_o is

$$\frac{d^2x}{dt^2} + m\eta\frac{dx}{dt} + m\omega_o^2 x(t) = F(t) \tag{12}$$

is

$$x(t) = m\left[G_R(t, t_o)\partial_{t_o} x_o - x(t_o)\partial_{t_o} G_R(t, t_o)\right] + \int_{t_o}^{t} dt'\, G_R(t, t')F(t')\,. \tag{13}$$

At late times (in the presence of any infinitessimal damping) the initial conditions can be ignored.

Similarly for the first order equation:

$$\left[m\frac{d}{dt} + m\eta\right] v(t) = F(t)\,; \tag{14}$$

the general solution is

$$v(t) = mG_R(t, t_o)v(t_o) + \int_{t_o}^{t} dt'\, G_R(t, t')F(t')\,. \tag{15}$$

Problem 2. Green function of the Diffusion equation

Consider the homogeneous diffusion equation:

$$\partial_t n - D\nabla^2 n(t, \boldsymbol{r}) = 0\,. \tag{16}$$

The retarded Green function of the equation satisfies

$$\left[\partial_t - D\nabla^2\right] G(t\boldsymbol{r}|t_o\boldsymbol{r}_o) = \delta(t - t_o)\delta^3(\boldsymbol{r} - \boldsymbol{r}_o)\,. \tag{17}$$

with retarded boundary conditions.

(a) Write Eq. (17) in time and $\boldsymbol{k}$ by introducing the spatial Fourier transform

$$G(t, \boldsymbol{k}) \equiv \int d^3\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} G(t, \boldsymbol{r})\,, \tag{18}$$

and then determine the retarded Green function of the diffusion equation in $\boldsymbol{k}$ and time.

(b) Determine the retarded Green function in ω and $\boldsymbol{k}$, $G_R(\omega, \boldsymbol{k})$, by Fourier transforming Eq. (17) in time and space. Verify that if you perform the Fourier integral over ω that you get the result of part (a).

(c) By taking the spatial Fourier transform verify that

$$G_R(\tau, \boldsymbol{r}) = \theta(\tau)\frac{1}{\sqrt[3]{2\pi\sigma^2(\tau)}} \exp\left(-\frac{(\boldsymbol{r} - \boldsymbol{r}_o)^2}{2\sigma^2(\tau)}\right) \tag{19}$$

where $\sigma^2(t) = 2D\tau$ where $\tau = t - t_o$