Problem 1. Green theorem for first and second order equations and the initial value problem

First order: Consider a model first order equation for the velocity

\[ m \frac{dv}{dt} + m \eta v = 0 \]  

(1)

describing how a particle slows down.

(a) Determine the Green function for this equation, \textit{i.e.} find the causal function that satisfies

\[ \left[ m \frac{d}{dt} + m \eta \right] G_R(t) = \delta(t) \]  

(2)

using the direct method, and by fourier transforms.

(b) Show that the solution at time \( t \) satisfying the boundary conditions specified at \( t = t_o \) are

\[ v(t) = m G_R(t,t_o)v(t_o) \]  

(3)

This is normally how the Green function (propagator) is used in quantum mechanics. The Green function is used slightly differently for second order equations, since \( x \) and \( \dot{x} \) enter the game.

Second order: In class we showed that the electric potential can be determined from knowledge of the boundary value and the Green function. A very similar statement can be made about an initial value problem, \textit{i.e.} the solution at future times can be determined from the initial conditions and the Green function.

For definiteness we will take a harmonic oscillator with mass \( m \) and resonant frequency \( \omega_o \):

\[ m \frac{d^2x}{dt^2} + m\omega_o^2 x = 0 \]

The retarded Green function \( G(t|t_o) \) is the position \( x(t) \) of the harmonic oscillator at time \( t \) from an impulsive force at time \( t_o \). It is causal, meaning that it vanishes whenever \( t < t_o \), \textit{i.e.}

\[ \left( m \frac{d^2}{dt^2} + m\omega_o^2 \right) G_R(t|t_o) = \delta(t-t_o) \quad \text{and} \quad G_R(t,t_o) = 0 \quad \text{for} \quad t < t_o \]  

(4)

(a) Given the initial conditions for the oscillator, \( x(t_o) \) and \( \partial_{t_o} x(t_o) \), at time \( t_o \) show that the future value of the oscillator \( x(t) \) is given by the Wronskian of the Green function and the initial conditions

\[ x(t) = m \left[ G_R(t,t_o) \partial_{t_o} x_o - x(t_o) \partial_{t_o} G_R(t,t_o) \right] \quad t > t_o \]  

(5)

Hint use the EOM to prove Greens theorem, \textit{i.e.} that the wronskian of the Green function and the solution we are looking for satisfies

\[ \partial_{t_o} \left[ x(t_o) \left( m \partial_{t_o} G_R(t,t_o) \right) - \left( m \partial_{t_o} x(t_o) \right) G_R(t,t_o) \right] = x(t_o)\delta(t-t_o). \]  

(6)
Then use this result together with the fact that \( G_R \) satisfies retarded boundary conditions to prove Eq. (5) . We also tacitly assume that \( G_R(t, t_o) \) satisfies

\[
\left( m \frac{d^2}{dt^2} + m \omega_o^2 \right) G_R(t|t_o) = \delta(t - t_o) \quad \text{and} \quad G_R(t, t_o) = 0 \quad \text{for} \quad t < t_o \quad (7)
\]

which is true because the harmonic oscillator is self adjoint.

You could also proceed directly, showing that Eq. (5) satisfies the equations of motion

\[
\left( m \frac{d^2}{dt^2} + m \omega_o^2 \right) x(t) = 0 \quad (8)
\]

and the initial conditions,

\[
\lim_{t \to t_o} x(t) = x(t_o) \quad (9)
\]

\[
\lim_{t \to t_o} \frac{dx(t)}{dt} = \partial_t x(t_o) \quad (10)
\]

(b) Use the Green function for the undamped oscillator given in class to verify that you get the correct result for \( x(t) \) in terms of the initial conditions.

(c) Show that for the wave equation, \(-\Box G_R(t|t_o, x_o) = \delta(t - t_o)\delta^3(x - x_o)\), the appropriate generalization is

\[
u(t, x) = \frac{1}{c^2} \int d^3 x_o [G(t|x|t_o, x_o)\partial_{t_o}u(t_o, x_o) - u(t_o, x_o)\partial_{t_o}G(t|x|t_o, x_o)] \quad (11)\]

**Remark:** The results of this problem show that the general solution to the driven damped harmonic oscillator starting from some initial time moment \( t_o \) is

\[
\frac{d^2 x}{dt^2} + m\eta \frac{dx}{dt} + m \omega_o^2 x(t) = F(t) \quad (12)
\]

is

\[
x(t) = m [G_R(t, t_o)\partial_{t_o}x_o - x(t_o)\partial_{t_o}G_R(t, t_o)] + \int_{t_o}^{t} dt' G_R(t, t')F(t'). \quad (13)
\]

At late times (in the presence of any infinitesimal damping) the initial conditions can be ignored.

Similarly for the first order equation:

\[
\left[ m \frac{d}{dt} + m\eta \right] v(t) = F(t) ; \quad (14)
\]

the general solution is

\[
v(t) = m G_R(t, t_o)v(t_o) + \int_{t_o}^{t} dt' G_R(t, t')F(t'). \quad (15)
\]
Problem 2. Green function of the Diffusion equation

Consider the homogeneous diffusion equation:

$$\partial_t n - D \nabla^2 n(t, r) = 0. \quad (16)$$

The retarded Green function of the equation satisfies

$$[\partial_t - D \nabla^2] G(t|t_o, r_o) = \delta(t - t_o) \delta^3(r - r_o). \quad (17)$$

with retarded boundary conditions.

(a) Write Eq. (17) in time and $k$ by introducing the spatial Fourier transform

$$G(t, k) \equiv \int d^3r e^{-i k \cdot r} G(t, r), \quad (18)$$

and then determine the retarded Green function of the diffusion equation in $k$ and time.

(b) Determine the retarded Green function in $\omega$ and $k$, $G_R(\omega, k)$, by Fourier transforming Eq. (17) in time and space. Verify that if you perform the Fourier integral over $\omega$ that you get the result of part (a).

(c) By taking the spatial Fourier transform verify that

$$G_R(\tau, r) = \theta(\tau) \frac{1}{\sqrt{2 \pi \sigma^2(\tau)}} \exp \left(-\frac{(r - r_o)^2}{2\sigma^2(\tau)} \right) \quad (19)$$

where $\sigma^2(t) = 2D\tau$ where $\tau = t - t_o$. 