Problem 1. (Optional. In class. Good exam material) Radiation during perpendicular acceleration

Consider an ultrarelativistic particle of velocity β experiencing an acceleration $a_\perp$ perpendicular to the direction of motion. Here $a_\perp$ points along the x-axis and $\boldsymbol{\beta}$ points along the z-axis.

(a) Show that the energy radiated per retarded time is approximately

$$\frac{dW}{dTd\Omega} = \frac{e^2}{16\pi^2c^3}\frac{a_\perp^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \tag{1}$$

$$\simeq \frac{e^2}{2\pi^2c^3}\frac{a_\perp^2}{(1+(\gamma\theta)^2)^3}\left[1-\frac{4(\gamma\theta)^2\cos^2\phi}{(1+(\gamma\theta)^2)}\right] \tag{2}$$

In the first equality, I give the full answer without approximation, but I will only grade the second approximate result.

Hint, in working out this radiation pattern you might (as a start) show without approximation that

$$|\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a}|^2 = (1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2a^2 - (\boldsymbol{n}\cdot\boldsymbol{a})^2(1-\beta^2) \tag{3}$$

by using the "b(ac)-(ab)c" rule. Then select a coordinate system were

$$\boldsymbol{\beta} = (0,0,\beta) \tag{4}$$

$$\boldsymbol{a} = (a_\perp,0,0) \tag{5}$$

$$\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{6}$$

(b) Work in a ultra-relativistic approximation, and compute the total power by integrating over the solid angle (as done in class) to show that you obtain the appropriate relativistic Larmour result[1]

$$\frac{dW}{dT} = \text{come on} \ldots \text{ you know it} \ldots \text{ right?} \tag{7}$$

[1]
$$\frac{dW}{dT} = \frac{e^2}{4\pi}\frac{2}{3c^3}\gamma^4a_\perp^2$$