Electricity and Magnetism:

$$\nabla \cdot \vec{E} = \rho / \varepsilon_0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

$$-\nabla \times E = \frac{\partial B}{\partial t}$$

$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ = Speed of Light

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Units (use heavyside Lorentz units)

Coulomb Law $F = \frac{Q^2}{4\pi\varepsilon_0 r^2}$. Choose a set of units where $\varepsilon_0 = 1$ then $\mu_0 = \frac{1}{c^2}$

Formally define:

$$\bar{E} = \sqrt{\varepsilon_0}\, E \qquad \bar{Q} = Q / \sqrt{\varepsilon_0} \qquad \bar{\rho} = \rho / \sqrt{\varepsilon_0}$$

$$\bar{B} = \sqrt{\varepsilon_0}\, B$$

Then Eqs become

$$\nabla \cdot \bar{E} = \bar{\rho}$$

$$\nabla \times \bar{B} = \frac{1}{c^2}\left(j + \frac{\partial \bar{E}}{\partial t}\right)$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Then drop the bars. same as setting $\varepsilon_0 = 1$

- As we will see $\bar{E}$ and $\bar{B}$ are very much the same thing. Define

$E_{HL} = \bar{E}$ and $B_{HL} = c\bar{B}$, so that E_{HL} and B_{HL} have the same units:

$$E_{HL} = \bar{E} \qquad\qquad B_{HL} = c\bar{B}$$

$$= E_{mks}\sqrt{\varepsilon_0} \qquad\qquad = c\, B_{mks}\sqrt{\varepsilon_0} = \frac{B_{mks}}{\sqrt{\mu_0}}$$

Then the equations read:

$$\nabla \cdot \vec{E}_{HL} = \rho_{HL}$$

$$\vec{\nabla} \times \vec{B}_{HL} = \frac{\dot{j}_{HL}}{c} + \frac{1}{c}\frac{\partial}{\partial t} E_{HL}$$

$$\nabla \cdot B_{HL} = 0$$

$$-\nabla \times E_{HL} = \frac{1}{c}\frac{\partial}{\partial t} B_{HL}$$

We will use this and stop writing HL unless needed.

Remarks

① To convert from MKS to HL set $\varepsilon_0 = 1$ and arrange to multiply all magnetic fields by c:

Ex $\quad F = q\vec{E}_{mks} + \vec{V} \times \vec{B}_{mks}$

$$= q\,\vec{E}_{mks} + \frac{v}{c} \times (c\vec{B}_{mks})$$

$$= q\,\vec{E}_{HL} + \frac{\vec{V}}{c} \times \vec{B}_{HL}$$

Then for E+M equations in HL units we see structure

$$\nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{B} = \frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\nabla \cdot B = 0$$

$$-\nabla \times \vec{E} = \frac{\partial \vec{B}}{c\,\partial t}$$

$$\vec{F} = q\left(E + \frac{\vec{v}}{c} \times \vec{B}\right)$$

- Note every time derivative comes with $\frac{1}{c}$, i.e.

$$\frac{1}{c}\frac{\partial}{\partial t}$$

- Every velocity is measured in units of c. For example:

$$\frac{\vec{j}}{c} = \frac{\text{charge}}{\text{vol}} \times \frac{\text{velocity}}{c}$$

Thus if the system has a characteristic length L and characteristic thime scale T, then the solutions will change rather dramatically from when

$$\underbrace{\frac{L}{c} \ll T}_{} \quad \text{vs} \quad \underbrace{T \ll \frac{L}{c}}_{}$$

electrostatics
+ magneto statics
+ quasi-static
System very rapidly adjusts to changes in the charges

Radiation dominated.
takes a finite time for Light to propagate across the system

Classical Mechanics of Charged Particles

$$\frac{d\vec{p}}{dt} = \vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

- Consider the fields specified and solve for the motion of the particles
- This is not really E+M, its classical mechanics + special relativity, we want to find $\vec{E}$ + $\vec{B}$

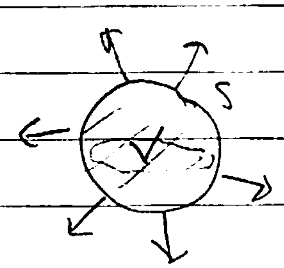
Fundamental Problem of Electrodynamics:

- Specify the charges and currents $\rho(\vec{x},t)$ and $\vec{j}(x,t)$ and solve for the fields. But respect continuity

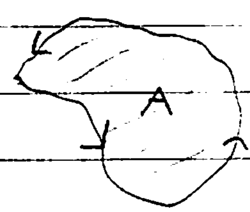
$$\partial_t \rho + \nabla \cdot \vec{j} = 0$$

Equations in integral form & Qualitative features

① Stokes theorem: $\displaystyle\int_{Vol} (\nabla \cdot \vec{V})\, d^3x = \int_{Area} d\vec{S} \cdot \vec{V}$



②



$$\int_{area} d\vec{S} \cdot (\nabla \times \vec{V}) = \int_{loop} \vec{V} \cdot d\vec{\ell}$$

Define

$$\Phi_B = \int_S \vec{B} \cdot d\vec{S} = \text{magnetic flux}$$

$$\Phi_E = \int_S \vec{E} \cdot d\vec{S} = \text{electric flux}$$

The maxwell eqs are

a) $\nabla \cdot E = \rho$

b) $\nabla \times \vec{B} = \frac{\vec{j}}{c} + \frac{1}{c}\partial_t \vec{E}$

c) $\nabla \cdot \vec{B} = 0$

d) $-\nabla \times E = \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

So for example:

$$\int_{Vol} \vec{\nabla} \cdot \vec{E} = \int_{Vol} \rho \, d^3x$$, this leads to -

Integral Forms:

a) $\int_{area} \vec{E} \cdot d\vec{a} = Q_{env}$ (Gauss Law)

b) $\oint \vec{B} \cdot d\vec{l} = \frac{I_0}{c} + \frac{1}{c}\frac{\partial \Phi_E}{\partial t}$ (Ampere's Law + maxwell correction)

c) $\int \vec{B} \cdot d\vec{S} = 0$ (No magnetic charge)

d) $-\oint \vec{E} \cdot d\vec{l} = \frac{1}{c}\frac{\partial \Phi_B}{\partial t}$ (Changing magnetic flux produces a backe Emf Faraday)

Lenz Law

Qualitative Features

a) $Q \rightarrow \vec{E}$ charges make e-fields (Gauss Law)

b) $\vec{J} \rightarrow \vec{B}$ moving charges (currents) make B-fields (Ampère)

d) Changing $\vec{B}$-fields make induced changing $\vec{E}$-fields

- Changing B-fields are caused by changing currents or accelerating charges

(Faraday)

b) Changing electric fields create changing B-fields (Maxwell correction), which in turn makes changing $\vec{E}$ etc,

This sets off a wave of light, where changing $\vec{E}$ makes changing $\vec{B}$ and vice-versa.

Remarks

(1) To get the whole process started you need accelerating charges:

Formula: (worth memorizing!)

$$P = \frac{2}{3}\left(\frac{e^2}{4\pi}\right)\frac{1}{c^3}a^2$$

P: power radiated by an accelerating charge

e^2: Charge^2

a: acceleration of charges

(2) We specify currents/charges and solve for fields. (The backreaction of the fields on the currents is not part of classical E&M. But is a part of quantum electrodynamics

Media — we will often study fields in media, where the currents and charges are a function of the fie[lds thro]ugh constitent relations, e.g. for a conductor

$$\vec{J} = \sigma \vec{E}$$

$\vec{J}$: current; σ: conductivity; $\vec{E}$: ~~electric field~~

- Such constituent relations depend on the medium, and are ~~usually~~ valid only when the wavelength is very long compared to micro scales
- For each medium, find a different set of equations to be solved, e.g. for a conductor

$$\nabla \times B = j/c + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\nabla \times B = \frac{\sigma}{c} E + \frac{1}{c}\frac{\partial E}{\partial t}$$

Now we solve this equation for the fields, with no reference to the currents.

- So when asked a E&M question about a medium (conductor, dielectric, topological insulator, ferromagnet), we should first be clear about what the constituent relations are!

→ Then solve the eqns for fields

→ Then you know the currents from constituent relation

$$j = \sigma E$$