Last Time - E&M in matter

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{j}{c} + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\partial_t B$$

need to specify currents.

When talking about fields in matter don't want to specify the matter currents:

$$\vec{j} = \vec{j}_{mat} + \vec{j}_{ext}$$

Need to guess the form of matter currents $j_{mat} = ?$

Key principles:

① Fields are Long wavelength & slow compared to micro - time scales

② Usually they are weak

③ Symmetry can be a clue:

Symmetry (Last Time Continued)

① Symmetry under rotations:

$$S \to \underset{\sim}{S} = S \qquad \text{(Scalars)}$$

$$V^i \to \underset{\sim}{V}^i = R^i{}_j V^j \qquad \text{(Vectors)}$$

$$T^{ij} \to \underset{\sim}{T}^{ij} = R^i{}_\ell R^j{}_m T^{\ell m} \qquad \text{(Tensors)}$$

② Parity $x \to -x$ (Eqs should be invouriant under parity)

③ Time reversal $t \to -t$ (Eqs should be inariant under time reversal)

- Dissipative coefficients are T-odd

$$\underbrace{M\frac{d^2x^i}{dt^2}}_{t\text{-even}} = -\underbrace{\eta}_{t\text{-odd}!}\ \overset{t\text{-odd}}{v^i}$$

Fourier Transforms:

$$E(\omega,k) = \int dt\, dr\, e^{i\omega t - i\vec{k}\cdot\vec{r}}\, E(\vec{t},\vec{r})$$

$$E(t,r) \longleftrightarrow E(\omega,k)$$

$$\partial_t E(t,r) \longleftrightarrow -i\omega E(\omega,k)$$

$$\partial_i E(t,r) \longleftrightarrow +ik_i(\omega,k)$$

(Last Time Continued)

Now we have from the conservation law

$$\partial_t \rho + \partial_i j^i = 0 \quad \longleftrightarrow \quad -i\omega\rho + ik \cdot j = 0$$

$$\rho = \frac{\vec{k} \cdot \vec{j}}{\omega}$$

Constituent Relations:

Now we need to specify $\vec{j}_{mat}$. Treat j_{mat} as an expansion in $\vec{E}$ field and its derivatives

- Electric field is weak keep only first order in E
- Also assume isotropic medium (no preferred axis)

$$\underbrace{\vec{j}_{mat}}_{\text{T odd}} = \underbrace{\sigma}_{\substack{\text{conductivity} \\ \text{T-odd}}} \underbrace{\vec{E}}_{\text{T-even}}$$

T-odd $\Leftarrow$ Dissipative process

For an insulator the conductivity is vanishingly small. We also will consider more terms

$$j^i = \underbrace{\cancelto{0}{\sigma} E^i}_{\text{T-odd}} + \underbrace{\chi\, \partial_t E^i}_{\text{T-even}} + \underbrace{\sigma_2\, \partial_t \partial_t E^i}_{\text{T-odd}} + \underbrace{\chi_3\, \partial_t^3 E^i}_{\text{Teven}} + \ldots$$

When the macroscopic time scales are long compared to the microscopic times, each higher term is suppressed

Constituent Relations (Continued)

Reason. Dimensional Analysis :

$[j] = \frac{q}{m^2}\frac{1}{S}$ $\quad$ S = seconds

m = meter

$[E] = \frac{q}{m^2}$ $\quad$ q = charge

So. $\quad$ Thus expect :

$[\chi] = 1$ $\quad$ $\chi \sim 1$

$[\sigma_2] = S$ $\quad$ $\sigma_2 \sim \tau_{micro}$ (or even less)

$[\chi_3] = S^2$ $\quad$ $\chi_3 \sim \tau_{micro}^2$

$[\sigma_4] = S^3$ $\quad$ $\sigma_4 \sim \tau_{micro}^3$

While for a macro-time scale $T \gg \tau_{micro}$

$$\partial_t E \sim \frac{1}{T} E \quad \text{and} \quad \partial_t^2 E \sim \frac{1}{T^2} E \quad \ldots$$

Thus

$$\vec{j} = \cancel{\sigma E}^{\,0} + \chi\, \partial_t E + \sigma_2\, \partial_t^2 E + \chi_3\, \partial_t^3 E$$

$$\sim 0 + \frac{E}{T} + \left(\frac{\tau_{mic}}{T}\right)\frac{E}{T} + \left(\frac{\tau_{mic}}{T}\right)^2 \frac{E}{T} + \ldots$$

Constituent Relation (Final)

Thus at lowest order in the gradient expansion

$$\vec{j} = \chi \partial_t \vec{E}$$

$$\boxed{\vec{j} = \partial_t \vec{P}}$$

$$\boxed{\vec{P} = \chi \vec{E}}$$

polarization vector

Linear isotropic media

T-odd   T-even   T-even

So we can work out the charge density

- From

$$\rho(\omega, k) = \frac{\vec{k} \cdot \vec{J}}{\omega} \quad \text{and} \quad \vec{j}(\omega, k) = -i\omega \vec{P} \Longleftrightarrow j = \partial_t \vec{P}$$

Find $\rho(\omega, k) = -i\vec{k} \cdot \vec{P}$ or $\boxed{\rho = -\vec{\nabla} \cdot \vec{P}}$

- Or could have used coordinate space

$$\partial_t \rho + \nabla \cdot j = 0$$

$$\partial_t \rho + \partial_i \partial_t P^i = 0$$

$$\partial_t (\rho + \partial_i P^i) = 0 \Rightarrow \boxed{\rho = -\partial_i P^i}$$

Constituent Relation in EOM

With this we get the Eqs of motion:

$$\nabla \cdot E = \rho_{mat} + \rho_{ext} \qquad \nabla \times E = 0$$

$$\nabla \cdot E = -\nabla \cdot P + \rho_{ext}$$

Now

$$\nabla \cdot (\vec{E} + \vec{P}) = \rho_{ext}$$

with $\vec{P} = \chi \vec{E}$ for linear isotropic matter

$$\nabla \times E = 0$$

So define $\boxed{\vec{D} = \vec{E} + \vec{P}}$ and find

$$\boxed{\left.\begin{aligned} \nabla \cdot \vec{D} &= \rho_{ext} \\ \nabla \times E &= 0 \end{aligned}\right\} \text{ eqs of macroscopic matter}}$$

Where $\vec{D} \equiv \vec{E} + \vec{P} \Rightarrow \underbrace{(1+\chi)}_{\equiv \varepsilon} \vec{E}$ for a linear medium isotropic medium

linear relation

$$\varepsilon = 1 + \chi$$

Non-linear ~~media~~ (isotropic) media

- only one derivative lots of powers of E

$$j = \chi \partial_t E + \cancel{\chi^{(2)} \partial_t E^2} + \chi^{(3)} \partial_t E^3 + \dots$$

parity odd

because

$\vec{j}$ would be irregular fcn of $\vec{E}$:

$$\vec{j} = \partial_t \vec{E} \chi + \chi^{(2)} \partial_t (\vec{E} \sqrt{E^2})$$

In fourier space

$$\vec{j} \to -i\omega \left[\chi E(\omega, k) + \chi^{(3)} (E^3)(\omega, k) + \dots \right]$$

So

$$\vec{j} = \partial_t \left[\chi \vec{E} + \chi^{(3)} E^3 + \dots \right]$$

$$\boxed{\vec{j} = \partial_t \vec{P}(E)}$$

← Defines the polarization vector

Still find

$$\boxed{\rho = -\vec{\nabla} \cdot \vec{P}}$$

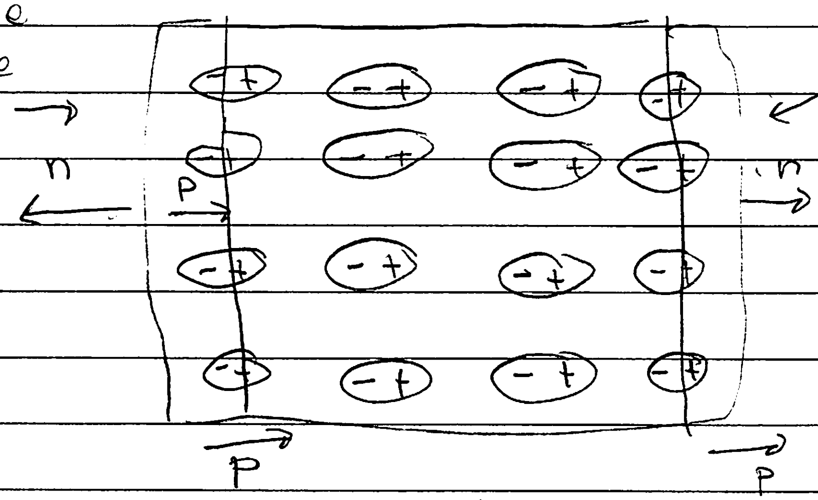
$$\underset{\text{T-odd}}{\vec{j}(\omega)} \xrightarrow[\omega \to 0]{} \underbrace{-i\omega}_{\text{T-odd}} \underbrace{\vec{P}}_{\text{T-even}}$$

Relation Between Constituent Relation + Dipole Picture

Now what is the relation between the current derivation and the picture of dipotes per volume:

Negative Charge here

There is positive Charge here



① First Lets calculate the surface charge.
The answer:

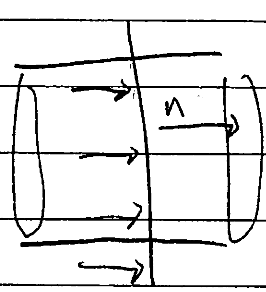
$$n \cdot \vec{P} = \sigma_{mat}$$

Proof: Previously we showed that

$$\vec{n} \cdot (\vec{E}_2 - \vec{E}_1) = \sigma$$

From

$$\nabla \cdot E = -\nabla \cdot P + \cancel{\sigma_f}^{\,0}$$

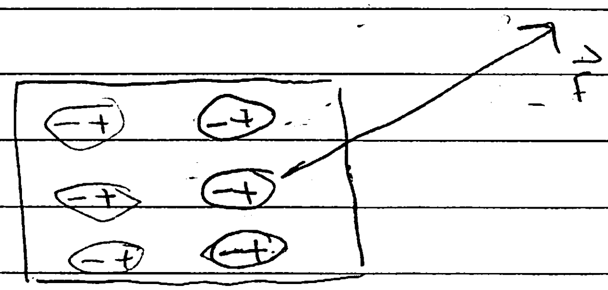


$$n \cdot (\vec{E}_2 - \vec{E}_1) = -n \cdot (\cancel{\vec{P}_2}^{\,0} - \vec{P}_1)$$

$$\boxed{\sigma_P = \vec{n} \cdot \vec{P}}$$

Relation to the Dipole Picture (continued)

(2)



What is the potential at $\vec{r}$:

$$\varphi = \int_V d^3x \, \frac{\rho_b(\vec{x})}{4\pi\,|\vec{r}-\vec{x}|} + \int_S d^2x \, \frac{\sigma_b(\vec{x})}{4\pi\,|\vec{r}-\vec{x}|}$$

$$\rho_b = -\nabla\cdot P = -\frac{\partial P^i}{\partial x^i} \quad \text{and} \quad \sigma_b = \vec{n}\cdot\vec{P}$$

So

$$\varphi = \underbrace{\int d^3x \, \frac{-\frac{\partial P^i(x)}{\partial x^i}}{4\pi\,|\vec{r}-\vec{x}|}}_{\text{integrate by parts}} + \int_S d^2x \, \frac{n\cdot P(x)}{4\pi\,|\vec{r}-\vec{x}|}$$

i.e.

$$\frac{-\frac{\partial P^i}{\partial x^i}}{4\pi\,|\vec{r}-\vec{x}|} = \underbrace{\frac{\partial}{\partial x^i}\left(\frac{-P^i}{4\pi\,|\vec{r}-\vec{x}|}\right)}_{\text{becomes a surface term}} + \underbrace{\frac{P^i\,(\vec{r}-\vec{x})_i}{4\pi\,|\vec{r}-\vec{x}|^3}}_{\text{volume terms}}$$

$$\varphi = \int_V \frac{p^i (r-x)_i}{4\pi |r-x|^3} + \cancel{\int \frac{-p^i n_i\, dS}{4\pi |\vec{r}-\vec{x}|}} + \cancel{\int \frac{p^i n_i\, dS}{4\pi |\vec{r}-\vec{x}|}}$$

$$\varphi = \underbrace{\int_V d^3\vec{x}\ \vec{P} \cdot \left[\frac{\overrightarrow{(r-x)}}{4\pi |\vec{r}-\vec{x}|^3} \right]}$$

Dipole potential $\propto \frac{1}{r^2}$

$$\vec{P} = \text{Dipole}^{\text{"acts like"}} \text{ moment } / \text{vol}$$