Last Time

Discussed Dielectric Matter

$$\nabla \cdot E = \rho_{mat} + \rho_f$$

$$\nabla \times B = \cancelto{\text{small}}{\frac{\vec{j}_{mat}}{c}} + \cancelto{\text{small}}{\frac{\vec{j}_f}{c}} + \cancelto{\text{small}}{\frac{1}{c}\partial_t \vec{E}}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\cancelto{\text{small}}{\frac{1}{c}\partial_t \vec{E}}$$

Make a $1/c$ expansion (crossing out terms with $1/c$

$$\nabla \cdot E = \rho_{mat} + \rho_f$$

$$\nabla \times E = 0$$

Then we looked at the current is a slow and constant ^in space ellectric field (but time dependent

$$\vec{j}_{mat} = \text{fcn of } E \text{ and its derivatives}$$

for insulator

polarizability

$$= \cancelto{0}{\sigma}\vec{E} + \partial_t \vec{P}(E) + \text{smaller higher derivs.}$$

- higher derivs are small since T_{micro}/T_{macro}
- Used dimensional analysis!

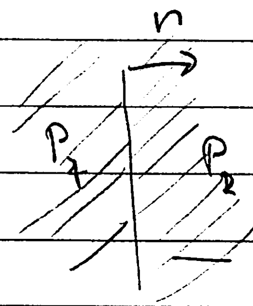
Last Time (Continued)

Then from current conservation

$$\partial_t \rho + \nabla\cdot \overbrace{j}^{\partial_t P} = 0$$

$$\partial_t \left[\rho + \nabla\cdot\vec{P} \right] = 0$$

$$\boxed{\rho_b = -\nabla\cdot P}$$



$$\boxed{\sigma_b = n(\vec{P}_2 - \vec{P}}$$

Now find:

$$\nabla\cdot E = -\nabla\cdot\vec{P} + \rho_{free}$$

$$\nabla\times E = 0$$

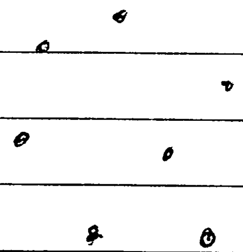
Thus:

$$\nabla\cdot(\overbrace{E+P}^{\equiv D}) = \rho_{free} \qquad D \equiv E+P$$

$$\nabla\times E = 0$$

$$\boxed{\begin{aligned} \nabla\cdot D &= \rho_f \\ \nabla\times E &= 0 \end{aligned}} \longrightarrow \quad \vec{E} = -\nabla\varphi$$

A model for the Polarizability:



Models are useful

→ Dimensions and Symmetries are right

→ Captures generic features

Last Time argued for Linear fields

$$\vec{J} = \cancel{\sigma \vec{E}} + \chi\, \partial_t \vec{E} + \sigma_2\, \partial_t^2 \vec{E} + \chi_3\, \partial_t^3 \vec{E} + \ldots$$

T-odd, T-even, T-odd dissipative, T-even

each term higher suppressed by ratios of micro/macro scales

Consider as model electrons harmonically bound to atoms, driven by harmonic E-field

$$m \frac{d^2 x}{dt^2} + m\eta \frac{dx}{dt} + m\omega_0^2 x = eE_0 e^{-i\omega t}$$

Drag - Todd, T-even

Micro timescales ≪ macro

A model for Polarizability (pg. 2)

<u>So</u> $\underset{\sim \frac{1}{T}}{\omega} \ll \underset{\frac{1}{\tau_{micro}}}{\omega_0}$ and η

Solving for steady state $x = x_0 e^{-i\omega t}$

$$\left[-m\omega^2 + m\eta(-i\omega) + m\omega_0^2 \right] x_0 e^{-i\omega t} = eE_0 e^{-i\omega t}$$

$$x(t) = \frac{(eE_0/m)\, e^{-i\omega t}}{-\omega^2 + \omega_0^2 - i\omega\eta}$$

Now expand for $\omega \ll \eta \ll \omega_0$ and compute current

assume dissipation is small

$$j(t) = \overset{\#/\mathrm{vol}}{n}\, e\, v(t) \qquad v(t) = -i\omega\, x(t)$$

$$j(t) = \frac{ne(eE_0/m)(-i\omega)\, e^{-i\omega t}}{-\omega^2 + \omega_0^2 - i\omega\eta}$$

expand

$$\simeq \frac{ne^2}{m\omega_0^2} \left[E(t)(-i\omega) + \frac{E(t)(-i\omega)^2 \eta}{\omega_0^2} + \frac{E(t)(-i\omega)^3}{\omega_0^3} + \dots \right.$$

involves drag!

$$j(t) \simeq \underbrace{\left[\frac{ne^2}{m\omega_0^2}\right]}_{=\chi\text{ - T-even}} \partial_t E + \underbrace{\left[\frac{ne^2}{m\omega_0^2}\frac{\eta}{\omega_0^2}\right]}_{=\sigma_2 \text{ T-odd}} \partial_t^2 E + \underbrace{\left[\frac{ne^2}{m\omega_0^2}\frac{1}{\omega_0^2}\right]}_{\chi_3 \text{ T-even}} \partial_t^3 E + \dots$$

Ar model for polarizability (pg. 3)

So we see the general structure emerge from a specific model.

We also estimate that

$$\boxed{\chi = \frac{ne^2}{m\omega_0^2}}$$ ← polarizability in this model

If material is densely packed:

$$n \sim \frac{1}{a_0^3} \qquad \omega_0 \sim \frac{\Delta E}{\hbar} \sim \frac{1}{\hbar}\left(\frac{e^2}{a_0}\right)$$

$$\chi \sim \frac{1}{a_0^3} \frac{e^2}{m \left(\frac{e^2}{\hbar a_0}\right)^2} \sim \frac{\hbar^2}{me^2 a_0} \sim 1$$

$$a_0 = \frac{me^2}{\hbar}$$

Find

$$\boxed{\chi \sim 1}$$

Solving Problems @ Dielectrics

$$\nabla \cdot D = \rho_f \qquad D = E + P$$

$$\nabla \times E = 0$$

Now what?

Solving E-statics @ dielectrics (pg. 2)

<u>Now we want to solve:</u>

① Need to specify $\vec{P}(E)$. For linear media

$$\vec{P} = \chi \vec{E} \quad \text{and} \quad \vec{D} = \vec{E} + \vec{P} = \overbrace{(1+\chi)}^{\equiv \varepsilon} \vec{E}$$

i.e. $\varepsilon = 1 + \chi$

Aside. In a crystal there are three axes. In this case the succeptibility is replaced by a tensor

$$P_i = \chi_{ij} E^j$$

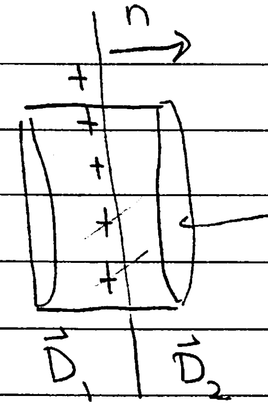
In a non-linear crystal we would have

$$P_i = \chi_{ij} E^j + \chi_{ijl} E^j E^l + \ldots$$

② Then we need boundary conditions

Solving E-statics @ Dielectrics (pg. 3)

- $\nabla \cdot D = \rho_f$

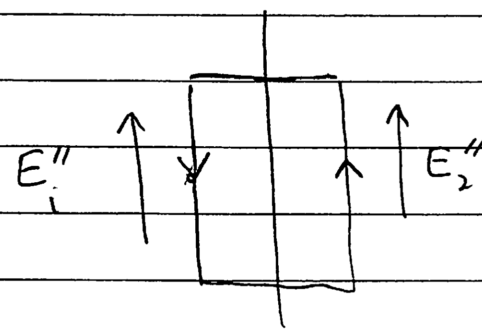


$$\boxed{\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma_f}$$

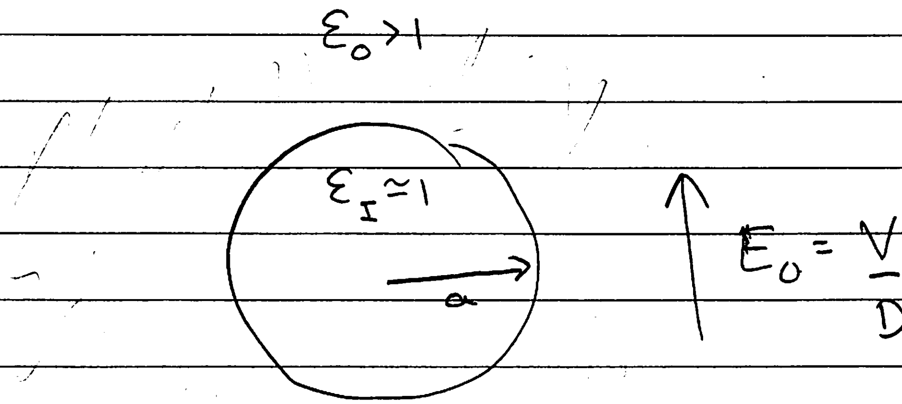
- $\nabla \times E = 0$

$$\boxed{E_2'' - E_1'' = 0}$$

$$\oint \vec{E} \cdot d\vec{l} = (E_2'' - E_1'')\, l = 0$$



Now lets solve a model Problem:



- A small dielectric sphere radius a, lies inbetween a capacitor of Voltage diff. V and separation D.

Dielectric Sphere pg. 2

Determine the potential and induced charge

Solution

- First write $\varphi = -E_0 z + \Phi$ and solve for Φ

$$\nabla \cdot D = \rho \implies -\varepsilon \nabla^2 \varphi = \cancel{\rho}^{\,0}$$

$$\nabla \times E = 0 \implies E = -\nabla \varphi$$

Find since $-\nabla^2(-E_0 z) = 0$ that:

$$-\nabla^2 \Phi = 0$$

Now B.C.:

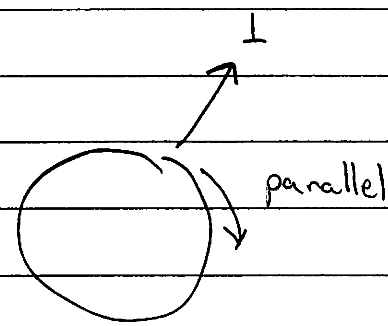
$$n \cdot D^{out} - n \cdot D^{in} = \cancel{\sigma_f}^{\,0}$$

$$E_2^{\parallel} - E_1^{\parallel} = 0$$

Means since $D = \varepsilon E$ than

$$\varepsilon_0 \frac{\partial \varphi_o}{\partial r} = \varepsilon_I \frac{\partial \varphi_I}{\partial r}$$

$$\frac{\partial \varphi_o}{\partial \theta} = \frac{\partial \varphi_I}{\partial \theta}$$



Dielectric Sphere pg. 3

The in solution

irregular at origin

$$\Phi_I = \sum_l \left(A_l \left(\frac{r}{a}\right)^l + \cancel{B_l \left(\frac{a}{r}\right)^{l+1}} \right) P_l(\cos\theta)$$

The out solution is

$$\Phi_o = \sum_l \left(\cancel{C_l \left(\frac{r}{a}\right)^l}^{\,0} + D_l \left(\frac{a}{r}\right)^{l+1} \right) P_l(\cos\theta)$$

Now require that

$$\left.\frac{\partial \Phi_I}{\partial \theta}\right|_{r=a} = \left.\frac{\partial \Phi_o}{\partial \theta}\right|_{r=0} \longleftarrow \text{for } \underline{\text{all}}\ \theta$$

This leads to the requirement $A_l = D_l$, So

$$\varphi = \begin{cases} -E_0 r \overbrace{\cos\theta}^{P_1} + \sum_l A_l \left(\frac{r}{a}\right)^l P_l \\ -E_0 r \overbrace{\cos\theta}^{P_1} + \sum_l A_l \left(\frac{a}{r}\right)^{l+1} \end{cases}$$

So from:

$$\varepsilon_0 \frac{\partial \varphi}{\partial r}\bigg|_{r=a} = \varepsilon_I \frac{\partial \varphi}{\partial r}\bigg|_{r=a}$$

Find:

$$\varepsilon_0\left(-E_0\cos\theta - \sum_l \frac{(l+1)}{a} A_l P_l(\cos\theta)\right) = \varepsilon_I\left(-E_0\cos\theta + \sum_l \frac{l}{a} A_l P_l(\cos\theta)\right)$$

Find:

$$-\varepsilon_0 E_0 - \varepsilon_0 \overbrace{\frac{(l+1)}{a}}^{2} A_l = -\varepsilon_I E_0 + \overbrace{l}^{1} \frac{A_l}{a} \qquad l=1$$

and

$$-\varepsilon_0 \frac{(l+1) A_l}{a} = \varepsilon_I \frac{l A_l}{a} \qquad l \neq 0$$

So

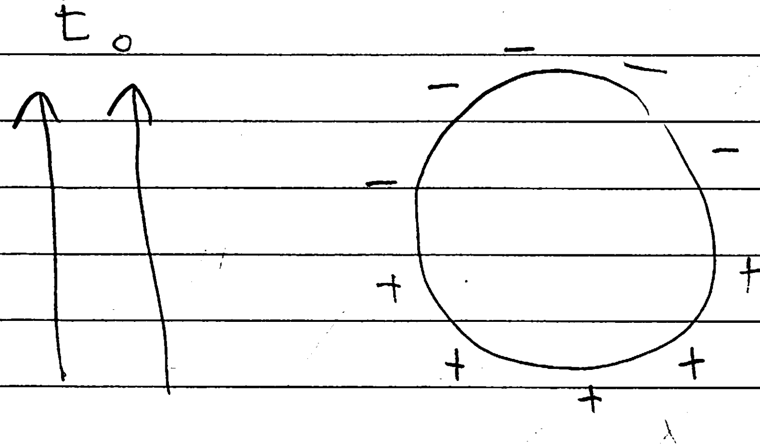
$$A_1 = \frac{(\varepsilon_I - \varepsilon_0)}{\varepsilon_I + 2\varepsilon_0} E_0 a \qquad l=1$$

$$A_l = 0 \qquad l \neq 1$$

$$\varphi = \begin{cases} -E_0 r\cos\theta + E_0 a\left(\frac{a}{r}\right)^2 \overbrace{\left(\frac{\varepsilon_I - \varepsilon_0}{\varepsilon_I + 2\varepsilon_0}\right)}^{\text{negative for } \varepsilon_I < \varepsilon_0} \cos\theta \\ -E_0 r\cos\theta \left(\frac{3\varepsilon_0}{\varepsilon_I + 2\varepsilon_0}\right) \end{cases}$$

(Dielectric Sphere pg. 5)

Picture



So the induced dipole moment is opposite the field

$$\sigma_p = \vec{n}\cdot(\vec{P}_2 - \vec{P}_1)$$

$$= (\varepsilon_o - 1)\, E_r^{out} - (\varepsilon_I - 1)\, E_r^{in}$$

$$E_r = -\frac{\partial \varphi}{\partial r}$$

$$\sigma = 3 \underbrace{\left(\frac{\varepsilon_I - \varepsilon_o}{\varepsilon_I + 2\varepsilon_o}\right)}_{\text{negative}} \cos\theta$$