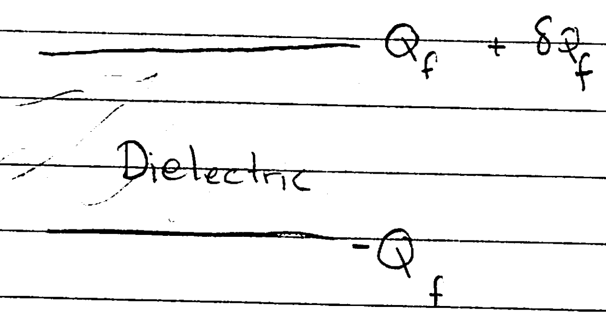
Energy in Dielectrics



The energy required to add dQ_f

$$\delta W = \int_V \delta\rho_f \cdot \varphi$$

External work required to charge capacitor including to polarize dielectric (see Griffiths)

$$= \int_V (\nabla \cdot \delta\vec{D})\,\varphi$$

$$= -\int_V \delta\vec{D} \cdot \vec{\nabla}\varphi$$

$$\delta W = \int_V \delta D^{(E)} \cdot \vec{E} \Rightarrow W = \int_V \int_0^D \vec{E}(D) \cdot d\vec{D}$$

For linear substance

$$\delta\vec{D} = \varepsilon(r)\,\vec{E}$$

$$\delta W = \int_V \varepsilon(r)\,\delta E \cdot E \Rightarrow W = \int_V \frac{1}{2}\varepsilon(r)\,\vec{E}^2(r)$$

Energy in Dielectrics Pg. 2

So

$$\boxed{W = \frac{1}{2}\int_V E \cdot \vec{D}}$$

For uniform substance (in thermo)

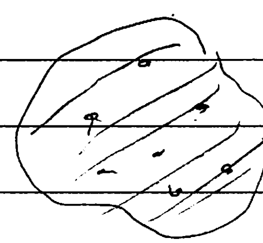
$$\boxed{dW = V\, \vec{E} \cdot d\vec{D}}$$

Forces and Stress Tensor

- Focus on material with $\varepsilon = \text{const}$ (Can treat piecewise const with this)

$$\vec{f} = \rho_f \vec{E} \qquad \text{(force per volume)}$$

Will show for linear media



$$F^j = -\int dS\, n_i T_E^{ij}$$

stress tensor (units) = force per area

$$T_E^{ij} = -E^i D^j + \frac{1}{2} E \cdot D\, \delta^{ij}$$

$$\boxed{T_E^{ij} = (-\varepsilon E^i E^j + \frac{1}{2}\varepsilon E^2)\delta^{ij}}$$

(Also works for ε not constant see book)

Same as vac by $\varepsilon \neq 1$

Prf

$$f^j = \rho_f E^j$$

$\nabla \times E = 0 \quad \partial_i E_j = \partial_j E_i$

$$= (\partial_i D^i) E^j$$

$$= \partial_i (D^i E^j) - D^i \partial_i E^j$$

$$= \partial_i (D^i E^j) - D^i \partial_j E^i$$

linear media
ε const

$$= \partial_i (D^i E^j) - \frac{1}{2} \partial_i (D \cdot E \, \delta^{ij})$$

$\partial_i D = \varepsilon \partial_i E$

$$f^j = \partial_i \underbrace{\left(D^i E^j - \frac{1}{2} D \cdot E \, \delta^{ij}\right)}_{= -T_E^{ij}}$$

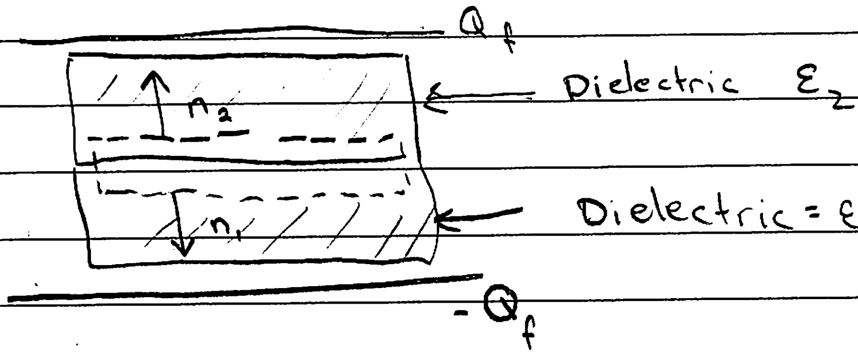
So

$$F^j = \int_V f^j = \int_V -\partial_i T^{ij}$$

$$F^j = -\int dS \, n_i T_E^{ij}$$

Force + Stress pg. 3

Problem



- Calculate the force per area on the interface. Method I; energy. Method 2 stress tensor

$$F^z = -\int_{\text{dased line}} dS \; n_i T^{ij}$$

$$= -A T_2^{zz} + A T_1^{zz}$$

Now

$$-T_2^{zz} = E^z D^z - \frac{1}{2} E \cdot D \, \delta^{zz} = +\frac{1}{2} E^z D^z = \frac{1}{2} \frac{(D_2^z)^2}{\varepsilon_2}$$

and similarly, $T_1^{zz} = \frac{1}{2}\frac{(D_1^z)^2}{\varepsilon_1}$. So

$$F^z = \frac{1}{2}\frac{(D_2^z)^2}{\varepsilon_2} - \frac{1}{2}\frac{(D_1^z)^2}{\varepsilon_1}$$

Using continuity of D at interface, $D_2 - D_1 = \cancel{\sigma_f}^{\,0}$. Ar at the metal plates $\vec{n}\cdot\vec{D} = \sigma_f$. Yielding

$$\boxed{F^z = \frac{1}{2}\sigma_f^2 \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)}$$