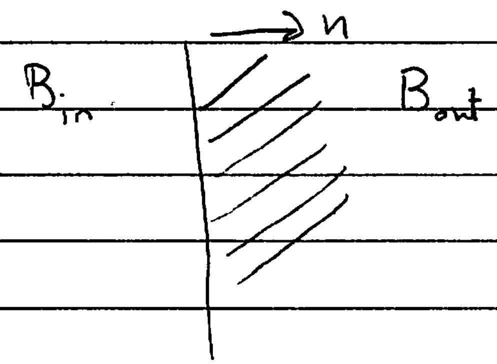
Last Times

$$\nabla \times B = j/c$$

$$\nabla \cdot B = 0$$

Wrote Down B.C



$$\vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) = \vec{K}/c$$

$$\vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) = 0$$

This can also be written for $\vec{A}$

$$-\nabla^2 \vec{A} = \vec{j}/c \qquad \vec{B} = \nabla \times \vec{A}$$

Question:

- What is the vector potential corresponding to a const $\vec{B}$ field?

Ans:

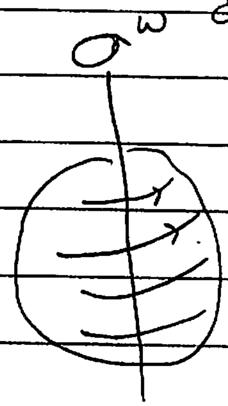
$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$$

Prf

$$(\nabla \times A)_i = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \frac{B^l r^m}{2}$$

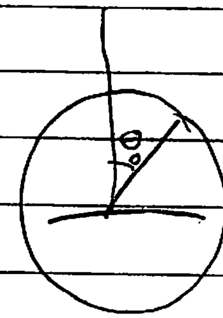
$$= \varepsilon_{ijk} \varepsilon_{lmk} \delta^m_j \frac{B^l}{2} = \overbrace{\varepsilon_{ijk} \varepsilon_{ljk}}^{2\delta_{il}} \frac{B^l}{2} = B_i$$

Example - Rotating Charged Sphere of Radius a, and Charged Q.



$$\vec{V} = \vec{\omega} \times \vec{x}$$

$$\vec{V} = \omega a \sin\theta_o \, \hat{\phi}$$



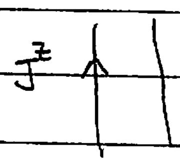
Now

$$\vec{J} = \rho \vec{V} \qquad \rho = \sigma \, \delta(r_o - a)$$

$$= \sigma \, \delta(r_o - a) \, \omega a \sin\theta_o \, \hat{\phi} \qquad \sigma = \frac{Q}{4\pi a^2}$$

Methods of Solution

- Direct use of $-\nabla^2 \vec{A} = \vec{J}$. In general complicated except for simple geometries, e.g. for 2D problems with azimuthal symmetry and no-z dependence, where only A^z is non-zero



$$-\nabla^2 A^z = J^z(\rho, \phi)$$

$$\left[-\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial}{\partial \phi^2} \right] A^z = J^z$$

- Magnetic scalar potential (next hour)

Rotating Sphere 2

Direct Integration

$$\vec{A}(\vec{r}) = \int d^3 r_0 \, \frac{\vec{j}(\vec{r}_0)}{4\pi |\vec{r} - \vec{r}_0|}$$

A good choice here:

$$(\star) \quad A(\vec{r}) = \int r_0^2 dr_0 \, d\Omega_0 \, \frac{\sigma \delta(r_0 - a) \, \omega a \sin\theta_0 / c}{4\pi |\vec{r} - \vec{r}_0|} \hat{\phi}_0$$

Now look outside sphere: $r > r_0$

$$\star\star \quad \frac{1}{4\pi |\vec{r} - \vec{r}_0|} = \sum_{lm} \frac{1}{(2l+1)} \frac{r_0^l}{r^{l+1}} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta_0, \phi_0)$$

where $\vec{r}_0 = (r_0 \sin\theta_0 \cos\phi_0, r_0 \sin\theta_0 \sin\phi_0, \cos\phi_0)$. Then look at

$$\sin\theta_0 \hat{\phi} = \underbrace{\sin\theta_0 \cos\phi_0 \, \hat{i}}_{\propto Y_{11} \text{ and } Y_{1-1}} + \underbrace{\sin\theta_0 \sin\phi_0 \, \hat{j}}_{\propto Y_{11} \text{ and } Y_{1,-1}}$$

Substituting Eq. ($\star\star$) into $\star$ we find integrals like:

$$\int d\Omega_0 \, Y^*_{lm}(\theta_0, \phi_0) (\sin\theta_0 \cos\phi_0) = 0 \quad \text{unless} \quad l = 1$$

Thus only $l = 1$ survives integrations $\int d\Omega_0$

Rotating Sphere 3

Comparison with the cartesian form

$$\frac{1}{4\pi|\vec{r}-\vec{r}_0|} = \frac{1}{(r^2+r_0^2-2\vec{r}\cdot\vec{r}_0)^{1/2}}$$

$$= \underbrace{\frac{1}{4\pi r}}_{l=0} + \underbrace{\frac{\vec{r}\cdot\vec{r}_0}{4\pi r^2}}_{l=1} + \underbrace{O(r_0^2)}_{l=2}$$

We see that inside the integral can replace

$$\frac{1}{4\pi|\vec{r}-\vec{r}_0|} \longrightarrow \frac{1}{4\pi}\frac{\vec{r}\cdot\vec{r}_0}{r^3}$$

So

$$\vec{A} = \frac{a^2\sigma}{c}\int d\Omega_0 \frac{1}{4\pi}\frac{\vec{r}\cdot\vec{r}_0}{r^3} \overbrace{(\vec{\omega}\times\vec{r}_0)}^{\omega a \sin\theta_0 \hat{\phi}}$$

So look at i-th component of this

$$\vec{r}\cdot\vec{r}_0\,(\vec{\omega}\times\vec{r}_0)^i = r^l (r_0)_l\, \varepsilon^{ijk}\omega_j (r_0)_k$$

So

$$A^i = \frac{a^2\sigma}{c} \frac{\varepsilon^{ijk}\omega_j r^l}{4\pi r^3} \underbrace{\int d\Omega_0 (r_0)_k (r_0)_l}$$

$$I = \frac{4\pi a^2}{3} \delta_{kl}$$

This integral follows as so:

$$\int d\Omega_0 (r_0)_k (r_0)_l = 0 \quad \text{for } k \neq l \qquad \int_{\text{sphere}} z\,y = 0$$

But $$\underbrace{\int (r_0)_1^2 \, d\Omega_0}_{\sim \langle x^2 \rangle} = \underbrace{\int (r_0)_2^2 \, d\Omega_0}_{\sim \langle y^2 \rangle} = \underbrace{\int (r_0)_3^2 \, d\Omega_0}_{\sim \langle z^2 \rangle}$$

So

$$\frac{1}{3}\int d\Omega_0 \underbrace{\vec{r}_0 \cdot \vec{r}_0}_{(r_0)_1^2 + (r_0)_2^2 + (r_0)_3^2 = a^2} = \frac{4\pi a^2}{3}$$

Rotating Sphere 5

Thus we find:

$$A^i = \frac{a^2\sigma}{c}\,\frac{\varepsilon^{ijk}}{4\pi r^3}\,\omega_j r^l \left(\frac{4}{3}\pi a^2 \delta_{kl}\right)$$

using $\sigma = Q/4\pi a^2$

$$A^i = \frac{1}{4\pi}\left[\left(\underbrace{\frac{Qa^2\,\vec{\omega}}{3c}}_{=\vec{m}}\right)\times \vec{r}\,\right]^i \Big/ r^3$$

magnetic dipole moment

So:

$$\vec{A} = \frac{1}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3} \qquad r > a \qquad \text{(outside)}$$

or $\vec{B} = (3(\vec{n}\cdot\vec{m})\,\vec{n} - \vec{m})/4\pi r^3$

Thus we see that the magnetic field outside the sphere is that of a magnetic dipole. An entirely similar calculation for $\vec{A}$ inside the sphere shows that:

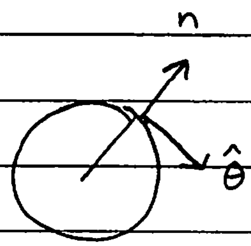
$$\vec{A}_{in} = \frac{1}{4\pi}\left(\frac{\vec{m}}{a^3}\right)\times\vec{r} \qquad r<a \qquad \text{(inside)}$$

This is the vector potential of a constant $\vec{B}$ field, $\vec{B}_{in} = \dfrac{2\,\vec{m}}{4\pi\,(a^3)}$

Rotating Sphere - Check BC

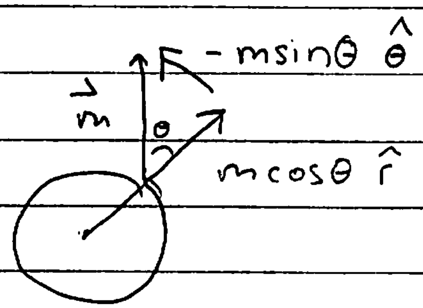
We will now check that the B.C. are satisfied.

$$\vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) = \frac{\vec{K}}{c}$$



$$(B_{out})_\theta - (B_{in})_\theta = \frac{K_\phi}{c}$$

Using:



this comes from $\vec{m}$ of $\dfrac{3(n\cdot\vec{m})\vec{n} - \vec{m}}{4\pi r^3}$

$$B^\theta_{out} = \frac{1}{4\pi a^3}(+m\sin\theta)$$

$$B^\theta_{in} = \frac{2(-m\sin\theta)}{4\pi a^3}$$

So

$$B^\theta_{out} - B^\theta_{in} = \frac{3m\sin\theta}{4\pi a^3}$$

$m = \dfrac{Qa^2\omega}{3c}$ $\quad \sigma = \dfrac{Q}{4\pi a^2}$

$$= \sigma \omega a \sin\theta$$

surface current ✓