Magnetic Matter

determined with electrostatics

- $$\nabla\times\vec{B} = \frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial\vec{E}^{(0)}}{\partial t}$$

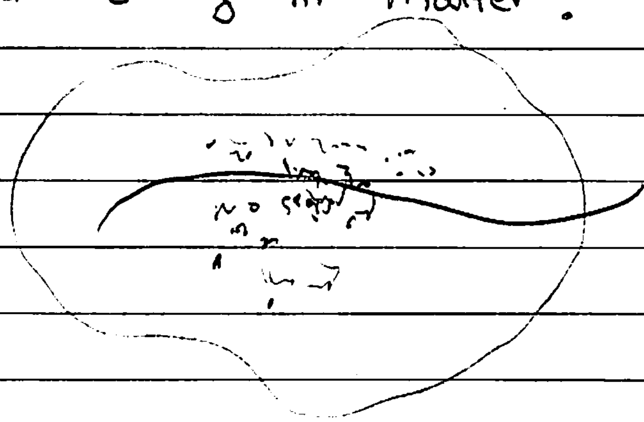
$$\nabla\cdot B = 0$$

For the moment set $E^{(0)} = 0$ (no net charge)

$$\nabla\times\vec{B} = \frac{j}{c} = \frac{j_{mat}}{c} + \frac{j_{free}}{c}$$

$$\nabla\cdot B = 0$$

What is $\vec{j}$ in matter?



long wavelength magnetic field

$L \gg l_{micro}$

Basic Principles

- Symmetry and long distance expansion
- very often weak

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Consider a magnetic field slowly varying in time and space. Write $\vec{j}$ as some general fcn of $\vec{B}$ and its derivatives

$$\vec{j} = \sigma_B \vec{B} + \chi_1 \partial_t \vec{B} + \chi_m (\nabla \times \vec{B}) + \ldots$$

+ higher derivatives
as before these are suppressed by powers of $\frac{l_{micro}}{L}$

Now recognize that symmetry forces many of these to be zero. Take the first term.

$$\underset{\substack{\text{P-odd} \\ \text{T-odd}}}{\vec{j}} = \underset{\substack{\text{P-odd} \\ \text{T-even}}}{\sigma_B} \; \underset{\substack{\text{P-even} \\ \text{T-odd}}}{\vec{B}}$$

The coefficients σ_B reflect the microscopic interactions. If the microscopic forces are invariant under parity.

$$\vec{x} \longrightarrow -\vec{x}$$

Then σ_B will be zero $\sigma_B = 0$

(But in other cases such as high temperatue electro weak plasma, which violates parity, this coefficient will not be zero, It will not be dissipative)

Unless otherwise specified we will assume parity invariance of microscopic forces and set $\sigma_B = 0$

$$\vec{j} = \cancel{\sigma_B \vec{B}} + \underbrace{\cancel{x_1 \partial_t \vec{B}}}_{\text{P-odd}} + \chi_m^B c(\nabla \times \vec{B}) + \ldots$$

So $\boxed{\frac{\vec{j}}{c} = \chi_m^B \vec{\nabla} \times \vec{B}} \implies \boxed{\frac{j}{c} = \nabla \times (\underbrace{\chi_m^B \vec{B}}_{=\vec{M}})}$ $= \text{magnetization}$

Now our equations of magneto-statics becomes

$$\nabla \times B = j_{mat} + \frac{j_{free}}{c} \qquad \nabla \cdot B = 0$$

$$\nabla \times \vec{B} = \nabla \times \vec{m} + \frac{j_{free}}{c}$$

And

$$\nabla \times \underbrace{(\vec{B} - \vec{m})}_{\equiv \vec{H}} = \frac{j_{free}}{c}$$

Leading to our eqs - Magneto Statics in matter

$$\boxed{\begin{aligned} \nabla \times \vec{H} &= \frac{j_{fr}}{c} \\ \nabla \cdot \vec{B} &= 0 \end{aligned}}$$

$$H[\vec{B}] = \vec{B} - \vec{M}(\vec{B})$$

$$\boxed{\vec{\nabla} \times \vec{M} = j_{mat}}$$

For a linear relation $\vec{M} = \chi^m_B \vec{B}$ and

$$\vec{H} = (1 - \chi^B_m)\vec{B}$$

$$\frac{1}{(1-\chi^B_m)} \vec{H} = \vec{B}$$

$$\boxed{\mu \vec{H} = \vec{B}}$$

$$\boxed{\mu = \frac{1}{1-\chi^B_m} = \text{permeability}}$$

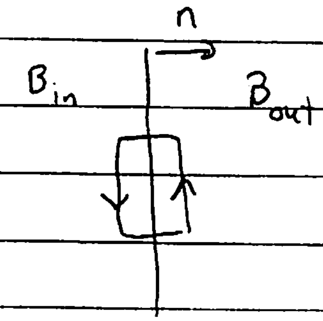
and we get a set of equations ; with $\vec{B} = \nabla \times \vec{A}$, for constant μ

$$\nabla \times \frac{(\nabla \times A)}{\mu} = j_{free}/c$$

This assumes a linear relation

$$\boxed{-\nabla^2 A = \mu \, j_{free}/c}$$

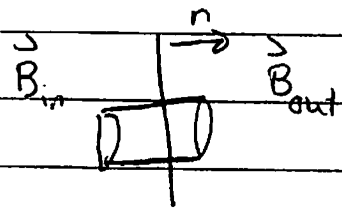
Boundary Conditions; previously found



$(\star)$

$$\nabla \times B = j/c \quad \Longrightarrow \quad \vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) = \frac{\vec{K}_{TOT}}{c}$$

$$\nabla \cdot B = 0$$



$$\vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) = 0$$

In Matter:

$$\nabla \times \vec{H} = j_{ext}/c \quad \Longrightarrow \quad \boxed{\begin{aligned} n \times (\vec{H}_{out} - \vec{H}_{in}) &= \frac{\vec{K}_{ext}}{c} \\ n \cdot (\vec{B}_{out} - B_{in}) &= 0 \end{aligned}} \quad (\star\star)$$

$$\nabla \cdot B = 0$$

Using $\vec{H} = \vec{B} - \vec{M}$ and $\vec{K}_{TOT} = \vec{K}_{mat} + \vec{K}_{ext}$
in Eq $(\star)$ and Eq $(\star\star)$;

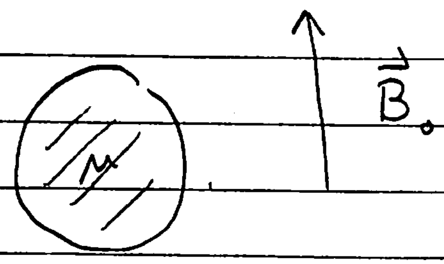
Find

$$\boxed{\vec{n} \times (\vec{M}_{out} - \vec{M}_{in}) = \vec{K}_{mat}/c}$$

in words the jump in the magnetization determines the current.

Magnetic Scalar Pot pg. 1

Example:



- The magnetic field induces a magnetic moment inside the material

$$\nabla \times H = \vec{J}_{ext}$$

$$B = \mu H$$

$$\nabla \cdot B = 0$$

In any current free region, $\nabla \times H = 0$
Then we can introduce a magnetic scalar potential, Ψ

$$\vec{H} = -\vec{\nabla}\Psi$$

Then inside the sphere:

$$\vec{\nabla} \cdot \left(\frac{-\nabla \Psi}{\mu} \right) = 0$$

assume $\mu = \text{const}$
(does not apply accross jump)

$$-\nabla^2 \Psi = 0$$

Similarly outside the sphere, $-\nabla^2 \Psi = 0$

Mag Scalar pg. 2

Then inside the sphere

$$\psi^{in} = \sum_{\ell} \left(A_\ell r^\ell + \cancel{\frac{B_\ell}{r^{\ell+1}}}^{\,0}\right) P_\ell(\cos\theta)$$

$$\psi^{out} = \sum_{\ell} \left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Inside, regularity forces $B = 0$

Outside, the fact that $B^{out} \xrightarrow[r\to\infty]{} B_0$, leads to

$$\psi_m \xrightarrow[r\to\infty]{} -B_0 r\cos\theta \qquad C_1 = -B_0 \quad \text{all other } 0$$

Now then we need boundary conditions

$$n \times (\vec{H}_{out} - \vec{H}_{in}) = \cancel{K_{free}/c}^{\,0}$$

$$\vec{n}\cdot(\vec{B}_{out} - \vec{B}_{in}) = 0 \Rightarrow \vec{n}\cdot\left(\vec{H}_{out} - \mu\vec{H}_{in}\right)$$

So

$$\vec{H} = -\nabla\psi_m = -\frac{\partial\psi}{\partial r}\hat{r} + -\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta}$$

Then

$$\psi_{out} = -B_0 r\cos\theta + \sum_{\ell} \frac{D_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Mag Scalar 3

So

$$H^r_{out} = +B_0 \cos\theta + \sum_l +(l+1)\frac{D_l}{r^{l+2}} P_l(\cos\theta)$$

$$H^\theta_{out} = -B_0 \sin\theta + \sum_l -\frac{D_l}{r^{l+2}} \frac{dP_l}{d\theta}(\cos\theta)$$

While

$$H^r_{in} = -\sum_l l A_l r^{l-1} P_l(\cos\theta)$$

$$H^\theta_{in} = -\sum_l A_l r^{l-1} \frac{dP_l}{d\theta}(\cos\theta)$$

Limitting ourselvete to $l=1$:

$$H^r_{out} = +B_0 \cos\theta + \frac{2D}{r^3}\cos\theta$$

$$H^\theta_{out} = -B_0 \sin\theta + \frac{D}{r^3}\sin\theta$$

In

$$H^r_{in} = -A\cos\theta$$

$$H^\theta_{in} = +A\sin\theta$$

Mag Scalar 5

Then from:

$$H^r_{out} - \mu H^r_{in} = 0 \Big|_{r=a} \Longrightarrow B_0 + \frac{2D}{a^3} + \mu A = 0$$

and

$$H^\theta_{out} - H^\theta_{in} = 0 \Big|_{r=a} \Longrightarrow -B_0 + \frac{D}{a^3} - A = 0$$

Solving gives:

$$D = B_0 a^3 \frac{(\mu-1)}{(\mu+2)} \qquad A = -\frac{3B_0}{2+\mu}$$

Further one can check that for $l \neq 1$ the eqs are trivially satisfied by $A_l = D_l = 0$.
Thus the full solution is

$$\psi^{in}_m = -\frac{3B_0}{2+\mu} r\cos\theta$$

$$\psi^{out}_m = -B_0 r\cos\theta + \overbrace{\frac{B_0 a^3}{r^2}\frac{(\mu-1)}{(\mu+2)}\cos\theta}^{\text{this takes the form of a dipole potential}}$$

or

$$\vec{H}^{in} = \frac{3B_0}{2+\mu}\hat{z}$$

$$\vec{H}^{out} = B_0\hat{z} + \left[3(\vec{n}\cdot\vec{m})\vec{n} - \vec{m}\right]/4\pi r^3$$

Mag Scalar 7 & B.C.

where,

$$\vec{m} = 4\pi B_0 a^3 \left(\frac{\mu-1}{\mu+2}\right).$$

Now lets check that Boundary Conditions Are Satisfied

The surface current is

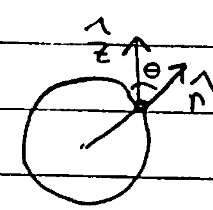
$$\vec{n} \times (\vec{M}_{out} - \vec{M}_{in}) = \vec{K}_{mat}/c$$

Now the magnetization outside the sphere is $= 0$ since we have no medium outside. Thus

$$-\vec{n} \times \vec{M}_{in} = \vec{K}_{mat}/c$$

Then $\vec{M} = (\mu-1)\vec{H}$, so

$$\frac{\vec{K}_{mat}}{c} = -\vec{n} \times \left[(\mu-1)\frac{3B_0}{2+\mu}\hat{z}\right]$$



$$\frac{\vec{K}_{mat}}{c} = (\mu-1)\frac{3B_0}{2+\mu}(-\vec{n} \times \hat{z})$$

see picture

$$\boxed{\frac{\vec{K}_{mat}}{c} = 3B_0\left(\frac{\mu-1}{\mu+2}\right)\sin\theta\,\hat{\phi}}$$

mag-Scalar B.C.

Thus, we see that the current distribution on the surface of the sphere:

$$\vec{K} \propto \sin\theta \, \hat{\phi},$$

is the same as for the rotating charged sphere, and thus the the induced magnetic fields in this case are the same (up to constant) as for the rotating charged sphere.