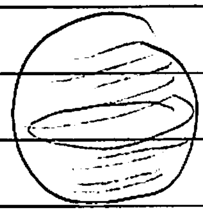
Math Discussion

Reduction of Tensor Integrals - A Useful/easy technique

$x = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$



- Three excercises to mastery

(1) $\int d\Omega_x \, x^i x^j = C\delta^{ij}$

$\underbrace{\int d\Omega_x \, \overbrace{x^i x_i}^{1}}_{4\pi} = C\cdot 3$

$\frac{4\pi}{3} = C$

(2) Consider an integral like this and reduce to scalar:

$$I^i = \int d\Omega \frac{x^i}{1+\vec{v}\cdot\vec{x}} = \int d\Omega \, x^i f(\vec{x}\cdot\vec{v})$$

Use rotational Symmetry to claim:

$$I^i = A(v)\,\hat{v}^i$$

Now dot both sides with $\hat{v}$

$$I^i\hat{v}_i = A(v) = \int d\Omega \, x^i \hat{v}_i \, f(\vec{x}\cdot\vec{v})$$

So now we are free to take v along z-axis

$$A(v) = \int d\Omega \cos\theta \, f(v\cos\theta)$$

$$= 2\pi \int_{-1}^{1} d(\cos\theta) \frac{\cos\theta}{1 + v\cos\theta}$$

So $I^i = A(v)\,\hat{v}^i$:

(3) Consider an integral - Excercise #3

$$I^{ij} = \int d\Omega \, x^i x^j f(\vec{x}\cdot\vec{v})$$

Reduce this integral to two scalars:

$$I_1 = \int d\Omega \cos^2\theta \, f(v\cos\theta)$$

$$I_2 = \int d\Omega \, f(v\cos\theta)$$

or Better use I_1 + $I_3 \equiv \int d\Omega \underbrace{\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)}_{P_2(\cos\theta)} f(v\cos\theta)$

Solution

$$I^{ij} = C(v)\,\delta^{ij} + D(v)\,\hat{v}^i v^j$$

Better

$$I^{ij} = \frac{1}{3}A(v)\,\delta^{ij} + B(v)\underbrace{\left(\hat{v}^i\hat{v}^j - \frac{1}{3}\delta^{ij}\right)}_{\text{Symmetric traceless}}$$

Solution

$$I^{ij} = \frac{1}{3} A(v)\, \delta^{ij} + B(v) \left[\hat{v}^i \hat{v}^j - \frac{1}{3}\delta^{ij} \right]$$

Taking trace

$$I^i{}_i = A(v)$$

$$= \int d\Omega\, \vec{x}\cdot\vec{x}\, f(\vec{x}\cdot\vec{v})$$

$$I_1 = A(v) = \int d\Omega\, f(v\cos\theta)$$

dotting Both sides (w) $\hat{v}$

$$\hat{v}_i I^{ij} v_j = \frac{1}{3} A(v) + \frac{2}{3} B(v)$$

$$\int d\Omega\, \vec{x}\cdot\hat{v}\, \vec{x}\cdot\hat{v}\, f(v\cos\theta) = \frac{1}{3} A + \frac{2}{3} B(v)$$

$$I_2 = \frac{1}{3} \overset{I_1}{A} + \frac{2}{3} B$$

$$\frac{3}{2} I_2 - \frac{1}{2} I_1 = B$$

So

$$I^{ij} = \frac{1}{3} I_1(v)\, \delta^{ij} + \overbrace{\left(\frac{3}{2} I_2 - \frac{1}{2} I_1\right)}^{= I_3} \left(\hat{v}^i \hat{v}^j - \frac{1}{3}\delta^{ij}\right)$$

(4)

$$T^{ij} = \left[\int_p d^3p\, f(p)\, p^i p^j p^l p^m\right] \chi_{lm}$$

Constant symmetric traceless tensor

Show that: a scalar integral of f

$$T^{ij} = I\, \chi^{ij}$$

And determine I: Solution start by saying

$$I^{ijlm} = \int d^3p\, f(p)\, p^i p^j p^l p^m = C\left[\delta^{ij}\delta^{lm} + \delta^{il}\delta^{jm} + \delta^{im}\delta^{jl}\right]$$

Contracting all indicíces:

$$I^{i\ \ l}_{\ i\ \ l} = C\,[\overbrace{3\cdot 3 + 3 + 3}^{15}]$$

$$\frac{1}{15} I^{i\ \ l}_{\ i\ \ l} = C$$

So we have

$$C = \frac{1}{15}\int \overbrace{d^3p}^{4\pi p^2 dp} f(p)\,(p^2)^2$$

$$C = \frac{4\pi}{15}\int_0^\infty dp\, f(p)\, p^6$$

So

$$T^{ij} = C\left[\delta^{ij}\delta^{lm} + \delta^{il}\delta^{jm} + \delta^{im}\delta^{jl}\right] x_{lm}$$

$$= C\left[0 + x^{ij} + x^{ij}\right]$$

$$T^{ij} = 2C\, x^{ij}$$

So

$$\boxed{T^{ij} = \left[\frac{8\pi}{15}\int_0^\infty dp\, f(p)\, p^6\right] x^{ij}}$$