Last Time

$$\nabla \times B = \frac{j_{mat}}{c} + j_{ext}/c$$

$$\nabla \cdot B = 0$$

Then we wrote down a constituent relation

$$\frac{\vec{j}_{mat}}{c} = \underbrace{\cancel{\sigma_B \vec{B}}}_{\substack{\text{parity}\\ \text{odd}}} + \underbrace{\cancel{\chi\, \partial_t \vec{B}}}_{\substack{\text{parity}\\ \text{odd}}} + \chi_m^B\, \nabla \times B + \text{higher}$$

$$\frac{\vec{j}_{mat}}{c} = \chi_m^B\, \nabla \times B \implies \frac{j_{mat}}{c} = \nabla \times M$$

magnetization

$$M = \chi_m^B \vec{B}$$

Then

$$\nabla \times B = \nabla \times M + j_{ext}/c$$

$$\nabla \cdot B = 0$$

Or

$$\boxed{\begin{array}{c} \nabla \times (\underbrace{B - M}_{\equiv H}) = j_{ext}/c \\[1em] \nabla \cdot B = 0 \end{array}}$$

Last Time pg. 2

Usually expressed as $M(H)$ rather than B.

Using,

$$H = B - M$$

$$H = B - \chi_m^B B$$

$$\frac{1}{(1-\chi_m^B)} H = B$$

i.e.

$$\boxed{\mu H = B} \quad \text{where} \quad \boxed{\mu = \frac{1}{(1-\chi_m^B)} = \text{permeability}}$$

We also recall the defining relation

$$\nabla \times M = \frac{j_{mat}}{c}$$

Linear Magnetic Materials

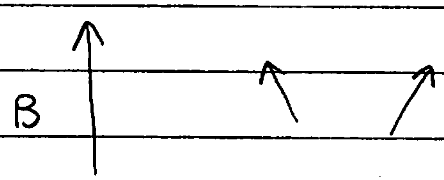
• Diamagnetic (oppose $\equiv$ dia) $\qquad \vec{M} = \chi_m^B \vec{B}$

$\chi_m^B < 0$ and $\mu < 1$ find $\chi_m^B \sim 10^{-5}$

Typicallly this is related to orbital motion of electrons, with all spins paired. The orbits change to oppose the change in flux

• Paramagnetic (same $\equiv$ para)

Typically related to spin aligning with the magnetic field



$\chi_m^B > 0$ and $\mu > 1$ $\qquad \chi_m^B \sim 10^{-5}$

Let us understand the order of magnitude of χ_m^B for diamagnetic and paramagnetic substaces

Dimensional Analysis of Linear Magnetic Substances & Ferromagnets

$$\vec{j} = \chi_m^B \, c \, \nabla \times \vec{B}$$

Then dimensions give

$$[\nabla \times B] = \frac{q}{m^2}\frac{1}{m}$$

$$[\vec{j}] = \frac{q}{m^2 s}$$

So, $[c\chi_m^B] = \frac{m}{s}$ so expect that $c\chi_m^B \sim V_{micro}$ ← naive dimensi

Where V_{micro} is the typical electron velocity.

In fact we can anticipate that since the forces which generate the currents, $\vec{F} = q\,(v/c) \times \vec{B}$ are small (since $V_{micro}/c \ll 1$) the currents are smaller than the naive dimension by v/c, ie

$$c\chi_m^B \sim V_{micro}\left(\frac{V_{micro}}{c}\right)$$

And thus

$$\chi_m^B \sim \left(\frac{V_{micro}}{c}\right)^2$$

Compare to the electric case $\chi_e \sim 1$

To estimate V_{micro}/c we recall the Bohr model. The Bohr model can be remembered by the Slogan "$\beta = \alpha$"

$$\beta = \frac{V_{bohr\,1s}}{c} = \alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137}$$

fine structure const

Also useful:

$$13.6\,eV = \frac{1}{2} PE = \frac{1}{2}\left(\frac{e^2}{4\pi a_0}\right) = KE = \frac{\hbar^2}{2ma_0^2} = \frac{1}{2}(mc^2)\,\alpha^2$$

Thus expect that

$$\chi_m^B \sim \alpha^2 \sim 10^{-5}$$

This works fine for linear substances. For Ferromagnetic materials, the magnetization can be much larger, and is usually non-linear

$$\vec{B} = \mu(H)\,\vec{H}$$

can be like 10^3

Why? Because Ferromagnetic Substances involve all the atoms working cooperatively even in the absence of exteral fields.

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

The spins tend to align in ferromagnetic substances because if the spin wave-fcn is symmetric, then the spatial wave-fcn can be anti-symmetric minimizing the coulomb emergy. This is a much larger effect (by $(v/c)^2$) than the dipole-dipole interaction, which would cause the the spins to anti-align.

In real ferromagnets the domains grow until the magnetic interaction competes with the short range coulomb interaction

Non-linear magnetic material & Hard Ferromagnets

In ferromagnets the induced magnetization depends non-linearly on $\vec{B}$. Assume $\vec{B}$ only vector

$$\vec{j}_{mat}(k) = \overbrace{\vec{B}(k)\left(\cancel{\sigma_B} + \cancel{\sigma_1} B^2(k) + \cancel{\sigma_2}(B^2(k))^2 + \dots\right)}^{\text{parity odd (thow away)}}$$

$$+ i\vec{k}\times B(\vec{k})\left(\chi_m^B + C_1 B^2(k) + C_2 (B^2(k))^2 + \dots\right)$$

$$+ \text{ higher derivs}$$

Thus, reasonably generally one finds a constituitive relation

$$\frac{\vec{j}_{mat}}{c} = \nabla\times\vec{M}(\vec{B})$$

← a nonlienear fuction of

after fourier transforming back to coordinate space. Thus the macroscopic equations for magnetostatics read

$$\nabla\times B = \nabla\times M(B) + j_{ext}/c$$

$$\nabla\cdot B = 0$$

$M(B)$ needs to be specified and generally gives rise to very non-linear equations

Hard Ferromagnets pg. 2

One case that can be handed is that of <u>hard ferromagnets</u> where $M(x)$ is a fixed function of space

$$\frac{j(x)}{c} = \nabla \times \vec{M}(x)$$

The boundary conditions still apply, namely

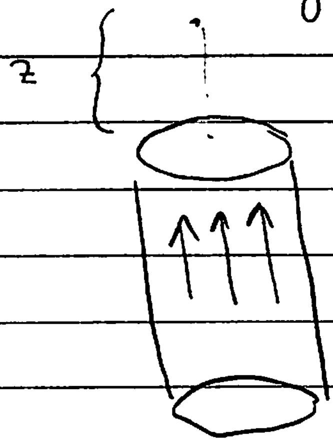
$$n \times (\vec{H}^{out} - \vec{H}^{in}) = \vec{K}_{free}/c$$

$$\vec{n} \cdot (\vec{B}^{out} - \vec{B}^{in}) = 0$$

$$n \times (\vec{m}_2 - \vec{m}_1) = \vec{K}_{mat}/c$$

Example

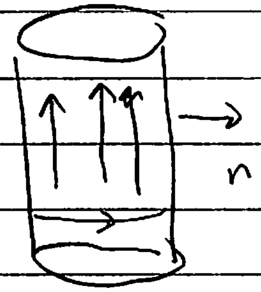
- A uniformly magnetized rod of height h, radius a, and magnetization $\vec{M} = M_0 \hat{z}$. Determine the magnetic field on axis.



$$\underset{=}{J_c} = \nabla \times M = 0 \quad \text{inside and out}$$

But we have boundary conditions which give a surface current

Using

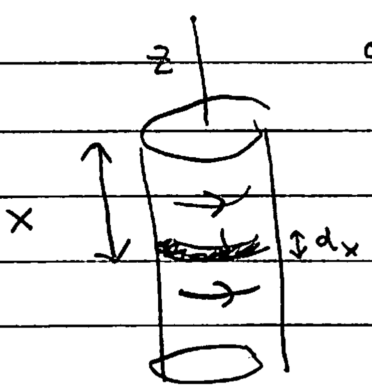


$$n \times (\cancel{m^{out}} - m^{in}) = \vec{K}_{mat}/c$$

$$-\vec{n} \times (M_0 \hat{z}) = \vec{K}_{mat}/c$$

$$M_0 \hat{\phi} = \vec{K}_{mat}/c$$

So we find a cylinder of current. From a Ring of of width dx



$$dB_z = \frac{dI}{2c} \frac{a^2}{(a^2 + (z+x)^2)^{3/2}}$$

$$\underset{=}{\frac{dI}{c}} = \frac{K\,dx}{c} = M_0\,dx$$

Example pg.2

$$B_z = \int_0^h dx \, \frac{M}{2} \frac{a^2}{((z+x)^2 + a^2)^{3/2}}$$

$$B_z = \frac{M}{2} \left[\frac{(h+z)}{(a^2 + (h+z)^2)^{1/2}} - \frac{z}{(a^2 + z^2)^{1/2}} \right]$$

Picture

$$B_z = \frac{M}{2} \left[\cos\theta_1 - \cos\theta_2 \right]$$