Maxwell Equations + Induction + Energy in Mag fields

\[ \nabla \cdot E = \rho_{\text{mat}} + \rho_{\text{ext}} \]

\[ \nabla \times B = \frac{j_{\text{mat}} + j_{\text{ext}}}{c} + \frac{1}{c} \frac{\partial}{\partial t} E \]

\[ \nabla \cdot B = 0 \]

\[-\nabla \times E = \frac{1}{c} \frac{\partial}{\partial t} B \]

Then, for the material current we write:

\[ j_{\text{mat}} = \frac{\sigma}{c} \frac{\partial}{\partial t} E + \frac{1}{c} \frac{\partial}{\partial t} P + \nabla \times M \]

Then with continuity, \( \rho_{\text{mat}} = -\nabla \cdot P \), \( D = E + P \), \( H = \frac{B}{c} - \frac{M}{c} \), find

\[ \nabla \cdot D = \rho_{\text{ext}} + \frac{1}{c} \frac{\partial}{\partial t} E + P \]

\[ \nabla \times H = \frac{j_{\text{ext}}}{c} + \frac{1}{c} \frac{\partial}{\partial t} D \] \( \leftarrow \) maxwell eqs

\[ \nabla \cdot B = 0 \]

\[-\nabla \times E = \frac{1}{c} \frac{\partial}{\partial t} B \]
Then we expand in powers of $c$

**Electrostatics**

$$\nabla \cdot D^{(0)} = \rho_{\text{ext}}$$

$$\nabla \times E^{(0)} = 0$$

**Magnetostatics:**

$$\nabla \times H^{(1)} = \mathbf{j}_{\text{ext}} / c + \frac{1}{c} \partial_t D^{(0)}$$

$$\nabla \cdot B^{(1)} = 0$$

**Induced Electric fields / Back Emf**

$$\nabla \cdot D^{(2)} = 0$$

$$-\nabla \times E^{(2)} = \frac{1}{c} \partial_t B^{(1)}$$

could call $E^{(2)} = E^{\text{ind}}$

Want to compute the energy stored in magnetic field

Back Emf

Imagine slowly increasing the current
changing the current makes a changing magnetic field inducing a Back Emf

The work the battery does to increase the current is the energy stored in the fields
Take \( \text{magneto statics}, \) i.e. \( D(0) = 0 \)

\[
\nabla \times H = -j, \\
\n\nabla \cdot B = 0
\]

And

\[
-\nabla \times E^{\text{ind}} = \frac{1}{c} \frac{d}{dt} B
\]

Then the work by battery is

\[
\Delta U = \frac{\Delta W_{\text{batt}}}{s_t} - \int_{s_t} \nabla \cdot \delta E^{\text{ind}} \cdot s
\]

\[
= -\int (\nabla \times H) \cdot c \delta E^{\text{ind}}
\]

\[
= -\int \nabla \cdot c \nabla \times \delta E^{\text{ind}}
\]

\[
\Delta U = \int \nabla \cdot c \nabla \times \delta E^{\text{ind}}
\]

\[
\Delta U = \int \nabla \cdot \delta B
\]

\[
\Delta U = \int \nabla \times \delta B
\]
Then for linear media \( \delta B = \mu \delta H \)

\[
U = \frac{1}{2} \int_2 \frac{H^2}{\mu} d^3x = U
\]

These equations are often expressed in terms of \( J \) and \( \vec{A} \) rather than \( \vec{B} \).

Indeed,

\[
\delta U = \int_2 \frac{\vec{H} \cdot \delta \vec{B}}{\mu} \quad \delta E^{ind} = -1 \frac{\varepsilon_0 \vec{E} \cdot \delta \vec{A}}{c}
\]

By parts (no minus because cross prod)

\[
\delta U = \int_2 \frac{\delta \vec{H} \cdot \vec{A}}{c} \quad \delta U = \int_2 \frac{\nabla \times \vec{H} \cdot \delta \vec{A}}{c}
\]

For linear media \( \delta A \propto \mu \delta J \)

\[
U = \frac{1}{2} \int_2 \frac{\vec{J} \cdot \vec{A}}{c}
\]
Inductance in Wires

1. \( U_B = \frac{1}{2} \oint \vec{J} \cdot \vec{A} \, d^3x = \frac{1}{2} \oint \vec{H} \cdot \vec{B} \)

   \( U_B \) is a property of state

2. \( SU_B = \oint \frac{1}{c} \cdot \delta \vec{A} \)

For a set of wires: \( \oint \vec{J} \cdot d^3x = \oint I dl \)

Then find

\( U = \frac{1}{2} \oint I_a \Phi_a \quad \Phi_a = \oint \vec{A} \cdot dl = \oint \vec{B} \cdot d\vec{A} \)

\( SU = \oint \frac{1}{c} \delta \Phi_a \quad = \) flux through \( a \)-th loop

Note that, \( \delta \vec{A}(x) = \rho \int \frac{\delta I(x_0)}{\sqrt{4\pi |x - x_0|}} \)

\( U_B = \frac{\mu}{2} \int \int d^3x \int d^3x_0 \cdot \frac{\delta}{c} \cdot \frac{\delta}{c} (x) \cdot \frac{I(x)}{c} \)

\( \left[ \frac{4\pi |x - x_0|}{x - x_0} \right] \)
So for a set of wires

\[ U_b = \frac{1}{2} M_{ab} I_b \]

& \quad \text{inductance matrix}

\[ M_{11} \text{ is the self inductance of the first loop} \]

\[ M_{12} \text{ is the mutual inductance between the 1st and 2nd} \]

Then since \[ U_b = \frac{1}{2} I_a \Phi_a \]

\[ \Phi_a = M_{ab} I_b \quad \text{back emf} \]

And for any circuit \[ E_a = -\frac{1}{2} \frac{d}{dt} \Phi_B = -M_{ab} \frac{dI_b}{dt} \]
Problem on Mutual Inductance and Force

- Compute the mutual inductance of a ring and a long straight wire.

\[ R \quad \Theta \quad I_1 \]

\[ D \quad \downarrow \quad I_2 \]

Solution

Current in wire one

\[ U_{12} = \oint \frac{I_2}{c} \cdot \vec{A}_2 \]

\[ = \frac{I_1}{c} \int \vec{A}_2 \cdot d\vec{l}_1 = \frac{I_1}{c} \int \vec{B}_2 \cdot d\vec{l}_1 \]

Then the field from the wire is

\[ \vec{B}_2 = \frac{I_2}{2\pi \rho} \hat{\phi} \]

So we need to integrate this field from the wire over the area of the ring.
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points out given by circulation of \( I_1 \)

\[ \oint \mathbf{l} \cdot \mathbf{d} \mathbf{a} = \frac{I_1}{c} \frac{2 \left( R^2 - (p - D)^2 \right)^{1/2}}{2 \pi p} \] points out

We have

\[ \mathbf{U}_{12} = \int_{D-R}^{D+R} dp \frac{I_1 I_2}{c^2} \frac{2 \left( R^2 - (p - D)^2 \right)^{1/2}}{2 \pi p} \]

\[ \mathbf{U}_{12} = \frac{I_1 I_2}{c^2} \left[ D - \sqrt{D^2 - R^2} \right] \]

So \[ \mathbf{M}_{12} = \frac{1}{c^2} \left( D - \sqrt{D^2 - R^2} \right) \]

Then we might want to compute the force between the ring and the wire. To do this we ask about the change in \( \mathbf{U}_{B} \) as the distance between the ring and the wire is changed:

\[ \Delta \mathbf{U}_{B} = \int_a \mathbf{F} \cdot d\mathbf{a} + \Delta W_{\text{mech}} \] work done on system

\[ \Delta W_{\text{batt}} \] change in energy

Stored in fields work done by battery to keep current fixed

\[ \mathbf{F} = -\frac{\mathbf{E}_{\text{ring}}}{\text{applied}} \] force on ring mechanically
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\[ S_U = \frac{1}{2} I_a \delta M_{ab} I_b \]

\[ I_a \delta \Phi_a = I_a \delta M_{ab} I_b \]

So

\[ S_{U_b} - I_a \delta \Phi_a = -\frac{1}{2} I_a \delta M_{ab} I_b = -F \delta D \]

So

\[ F_x = \frac{+8 M_{ab} I_1 I_2}{8D \Delta} = \frac{I_1 I_2 (1 - \frac{D}{c^2})}{\sqrt{D^2 - R^2}} \]

\[ = -\frac{I_1 I_2}{c^2} \frac{(D - 1)}{\sqrt{D^2 - R^2}} \]

indicates an attractive force

i.e. force in negative x-direction

\[ \overrightarrow{F} = \frac{D/R}{I_1 I_2/c^2} \rightarrow \text{attractive force} \]