Maxwell Equations + Induction + Energy in Mag fields

$$\nabla \cdot E = \rho_{mat} + \rho_{ext}$$

$$\nabla \times B = \frac{j_{mat}}{c} + \frac{j_{ext}}{c} + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

Then for the material current we write

$$\frac{j_{mat}}{c} = \overset{0 \text{ for insulator}}{\cancel{\frac{\sigma}{c}\vec{E}}} + \frac{1}{c}\partial_t \vec{P} + \nabla \times \vec{m}$$

Then with continuity, $\rho_{mat} = -\nabla \cdot P$, $D \equiv E + P$, $H = \vec{B} - \vec{m}$, find

$$\boxed{\begin{aligned} \nabla \cdot D &= \rho_{ext} \\ \nabla \times H &= \frac{j_{ext}}{c} + \frac{1}{c}\partial_t \overbrace{\vec{D}}^{E+\vec{P}} \\ \nabla \cdot B &= 0 \\ -\nabla \times E &= \frac{1}{c}\partial_t \vec{B} \end{aligned}}$$

← maxwell eqs in simple matter

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Then we expand in powers of c

Electrostatics

$$\nabla \cdot D^{(0)} = \rho_{ext}$$

$$\nabla \times E^{(0)} = 0$$

Magnetostatics:

$$\nabla \times H^{(1)} = j_{ext}/c \quad + \frac{1}{c}\partial_t D^{(0)}$$

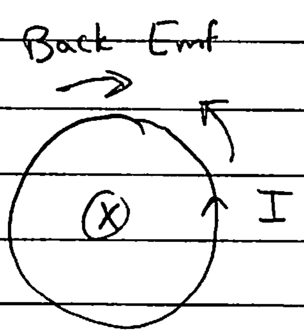
$$\nabla \cdot B^{(1)} = 0$$

Induced Electric fields / Back Emf

$$\nabla \cdot D^{(2)} = 0$$

$$-\nabla \times E^{(2)} = \frac{1}{c}\partial_t B^{(1)} \qquad \text{could call } E^{(2)} = E^{ind}$$

Want to compute the energy stored in magnetic field



Imagine slowly increasing the curre[nt]
changing the current makes a changing magnetic field inducing a Back Emf, The work the Battery does to increase the current is the energy stored in the fields

Take almost magneto statics, i.e. $D^{(0)} = 0$

$$\nabla \times H = j$$
$$\nabla \cdot B = 0$$

And

$$-\nabla \times E^{ind} = \frac{1}{c}\partial_t B$$

Then the work by battery is

$$\frac{\delta U}{\delta t} = \frac{\delta W_{batt}}{\delta t} = -\int \vec{j} \cdot \delta E^{ind} / \delta t$$

$-\vec{F} \cdot \vec{v}$

$$= -\int (\nabla \times H)\, c\, \delta E^{ind}$$

$\nabla \cdot (H \times E) = (\nabla \times \vec{H}) \cdot E - H \cdot \nabla \times \vec{E}$

$$= -\int_V \vec{H} \cdot c \nabla \times \delta E^{ind}$$

$$\frac{\delta U}{\delta t} = \int_V \vec{H} \cdot \frac{\delta \vec{B}}{\delta t}$$

$$\boxed{\delta U = \int_V \hat{H} \cdot \delta \vec{B}}$$

Then for linear media $\delta B = \mu \delta H$

$$U = \frac{1}{2}\int_V \frac{H^2}{\mu} = \boxed{\frac{1}{2}\int \vec{H}\cdot\vec{B}\, d^3x = U}$$

These equations are often expressed in terms of $\vec{J}$ and $\vec{A}$ rather than $\vec{B}$

Indead,

$$\delta U = \int_V \vec{H}\cdot\delta\vec{B} \qquad \delta E^{ind} = -\frac{1}{c}\partial_t \delta\vec{A} - \cancel{\nabla\phi}$$

$$\delta U = \int_V \vec{H}\cdot\nabla\times\delta\vec{A}$$

By parts (no minus) becaus cross prod

$$\delta U = \int \underbrace{\nabla\times H}_{\frac{\vec{J}}{c}} \cdot \delta\vec{A}$$

$$\boxed{\delta U_B = \int_V \frac{\vec{J}}{c}\cdot\delta\vec{A}}$$

For Linear media $\delta\vec{A} \propto \mu\,\delta\vec{J}$

$$\boxed{U = \frac{1}{2}\int \frac{\vec{J}}{c}\cdot\vec{A}}$$

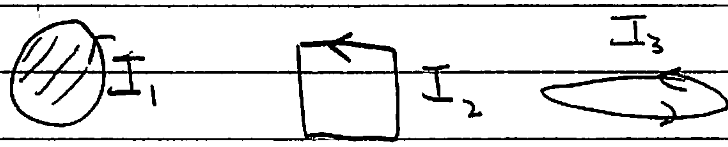
Inductance in Wires

① $U_B = \frac{1}{2}\int_V \vec{j}\cdot\vec{A}\, d^3x = \frac{1}{2}\int_V \vec{H}\cdot\vec{B}$

- U_B is a property of state

② $\delta U_B = \int_V \frac{\vec{j}}{c}\cdot\delta\vec{A}$

For a set of wires: $\vec{j}\, d^3x = I\, d\vec{l}$



Then find

summed over a = loops

① $U = \frac{1}{2}\frac{I_a}{c}\Phi_a$ $\qquad \Phi_a = \oint_{a\text{-th loop}} \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{A}$

② $\delta U = \frac{I_a}{c}\,\delta\Phi_a$ $\qquad$ = flux through a-th loop

Note that, $\vec{A}(x) = \mu\int_V \frac{j/c(x_0)}{4\pi|x-x_0|}$

$$U_B = \frac{\mu}{2}\int d^3x\, d^3x_0\, \frac{\vec{j}(x)/c\cdot\vec{j}(x_0)/c}{4\pi|x-\vec{x}_0|}$$

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So for a set of wires

$$U_B = \frac{1}{2} I_a M_{ab} I_b$$

inductance matrix

M_{11} is the self inductance of the first loop

M_{12} is the mutual inductance between the 1st + 2nd

Then since $U_B = \frac{1}{2} \frac{I_a}{c} \Phi_a$

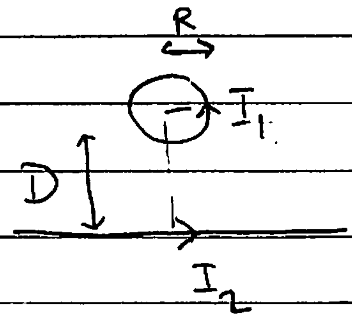
$$\frac{\Phi_a}{c} = M_{ab} I_b$$

back emf

And for any circuit $\mathcal{E}_a = -\frac{1}{c} \partial_t \Phi_B = -M_{ab} \frac{dI_b}{dt}$

Problem on Mutual Inductance & Force

- Compute the mutual inductance of a ring and a long straight wire



Solution

current in wire one

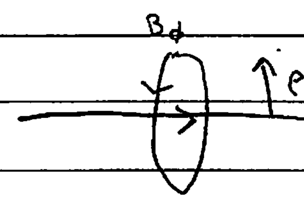
field from wire 2 at wire 1

$$U_{12} = \int_V \frac{\vec{J}_1}{c} \cdot \vec{A}_2$$

$$= \frac{I_1}{c} \int \vec{A}_2 \cdot d\vec{l}_1 = \frac{I_1}{c} \int \vec{B}_2 \cdot d\vec{a}_1$$

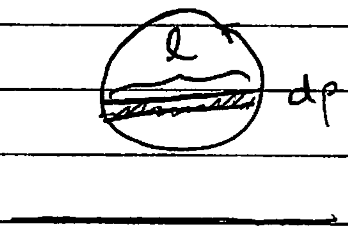
Then the field from the wire is

$$\vec{B}_2 = \frac{I_2/c}{2\pi\rho}\,\hat{\phi}$$



So we need to integrate this field from the wire over the area of the ring

Mutual Inductance & Force pg. 2



$$\underset{\text{points out}}{\vec{B}_2} \cdot \overset{\text{points out}}{\hat{n}}\, d\vec{a} = \frac{I_2/c}{2\pi\rho}\, \overbrace{2\left(R^2-(\rho-D)^2\right)^{1/2}}^{\ell \text{ (see picture)}}\, d\rho$$

given by circulation of I_1

We have

$$U_{12} = \int_{D-R}^{D+R} d\rho\, \frac{I_1 I_2}{c^2}\, \frac{2\left(R^2-(\rho-D)^2\right)^{1/2}}{2\pi\rho}$$

$$U_{12} = \frac{I_1 I_2}{c^2}\left[D - \sqrt{D^2-R^2}\right)$$

So $M_{12} = \frac{1}{c^2}\left(D - \sqrt{D^2-R^2}\right)$

Then we might want to compute the force between the ring and the wire. To do this we ask about the change in U_B, as the distance between the ring and the wire is changed:

$$\underset{\substack{\text{Change}\\\text{in energy}\\\text{Stored in fields}}}{\delta U_B} = \underbrace{I_a\, \delta\Phi_a}_{\delta W_{batt}} + \delta W_{mech}$$

δW_{batt}: work done by battery to keep current fixed

δW_{mech}: ~~work done~~ on system by mechanical forces $= -\vec{F}_{ring} \cdot \delta\vec{D}$

force on ring $\vec{F}_{ring} = -\vec{F}_{\text{applied mechanically}}$

Mutual Inductance and Force pg. 3

$$\delta U = \frac{1}{2} I_a \, \delta M_{ab} \, I_b$$

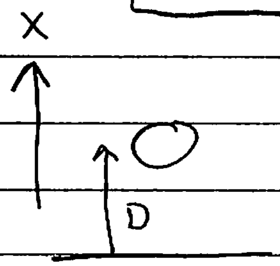
$$I_a \, \delta \Phi_a = I_a \, \delta M_{ab} \, I_b$$

So

$$\delta U_\beta - I_a \, \delta \Phi_a = -\frac{1}{2} I_a \, \delta M_{ab} \, I_b = -\vec{F} \, \delta \vec{D}$$

So

$$\boxed{F^x = + \frac{\delta M_{ab}}{\delta D} \frac{1}{2} I_a I_b} = \frac{I_1 I_2}{c^2} \left(1 - \frac{D}{\sqrt{D^2 - R^2}} \right)$$



$$= -\frac{I_1 I_2}{c^2} \left(\frac{D}{\sqrt{D^2 - R^2}} - 1 \right)$$

↑ indicates an attractive force i.e. force in negative x-direction