## Last Time

Talked about inductance

$$\nabla \cdot D = \rho$$
$$\nabla \times H = j_{ext}/c + \frac{1}{c}\partial_t \vec{D}$$

$$\nabla \cdot B = 0$$

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

Then

0: $\nabla \cdot D^{(0)} = \rho$

$\nabla \times E^{(0)} = 0$

1st $\nabla \times H^{(1)} = j_{ext}/c + \frac{1}{c}\partial_t \vec{D}^{(0)}$

$\nabla \cdot B^{(1)} = 0$

2nd $-\nabla \times E^{(2)} = \frac{1}{c}\partial_t B^{(1)}$

So concluded:

$$\delta U_B = \int_V H \cdot \delta B = \int_V \frac{\vec{j}}{c} \delta\vec{A}$$

Integrate $\delta B \propto \delta H$

$$U_B = \frac{1}{2}\int_V H \cdot B = \frac{1}{2}\int_V \vec{j} \cdot \vec{A}$$

Potentials pg. 1

Can also express in terms of potentials

① $\nabla \cdot E = \rho$

② $\nabla \times B = \vec{j}/c + \frac{1}{c}\partial_t \vec{E}$

③ $\nabla \cdot B = 0$

④ $-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$

So then

$$\vec{B} = \nabla \times \vec{A} \qquad \text{from ③}$$

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi \qquad \text{from ④}$$

Then from ①

$$\nabla \cdot \left(-\frac{1}{c}\partial_t \vec{A} - \nabla\varphi\right) = \rho \Rightarrow \boxed{-\nabla^2\varphi \quad \frac{1}{c}\partial_t(\nabla\cdot\vec{A}) = \rho}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi - \frac{1}{c}\frac{\partial}{\partial t}\left(+\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\vec{A}\right) = \rho$$

same

$$\boxed{-\Box\varphi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\varphi + \nabla\cdot\vec{A}\right) = \rho}$$

$$\Box = -\frac{1}{c^2}\frac{\partial}{\partial t^2} + \nabla^2 = \text{d'Alembertian}$$

And from ②

$$\underbrace{\nabla \times (\nabla \times \vec{A})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2 \vec{A}} = \vec{j}/c + \frac{1}{c}\partial_t\left(-\frac{1}{c}\partial_t \vec{A} - \nabla \varphi\right)$$

Then:

$$-\left(-\frac{1}{c^2}\partial_t^2 + \nabla^2\right)\vec{A} + \nabla\cdot\left(\frac{1}{c}\partial_t \varphi + \nabla\cdot\vec{A}\right) = j/c$$

i.e

$$\boxed{-\Box \vec{A} + \nabla\cdot\left(\frac{1}{c}\partial_t \varphi + \nabla\cdot\vec{A}\right) = \frac{j}{c}}$$

Then note that there is a constaint

$$\partial_t \rho + \nabla\cdot\vec{j} = 0$$

So there are only thee equations here, And we must specify a gauge condition in order to solve

① Coulomb Gauge:

$$\nabla \cdot \vec{A} = 0$$

Then

$$\boxed{\begin{aligned} -\nabla^2 \varphi &= \rho \\ -\Box \vec{A} &= \vec{j}/c + \frac{1}{c}\partial_t(-\nabla\varphi) \end{aligned}}$$

Often good for non-rel problems (Quasi-statics) And matter (ultra-relativistic plasma)

② Covariant Gauge:

$$\frac{1}{c}\partial_t \varphi + \nabla \cdot \vec{A} = 0$$

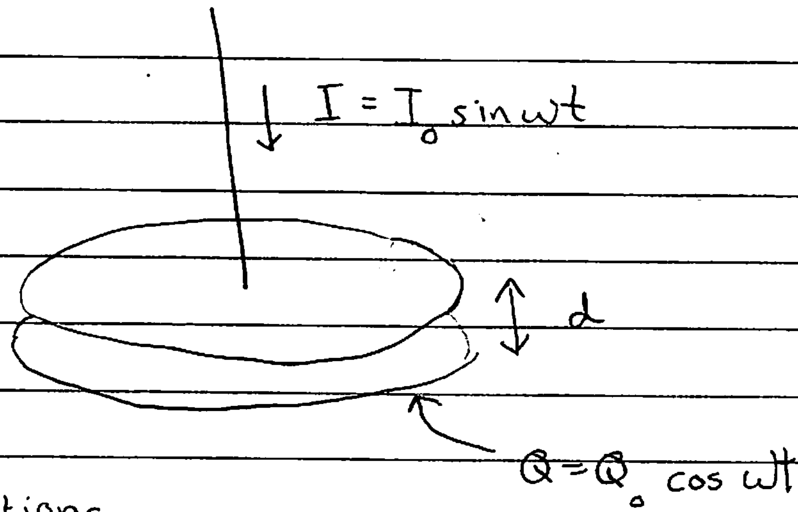
Then

$$\boxed{\begin{aligned} -\Box \varphi &= \rho \\ -\Box \vec{A} &= \vec{j}/c \end{aligned}}$$

Often a good choice for rel-problem with no preferred frame

Capacitor pg. 1 (dimensional analysis)

## An important example




## Questions

① What are the dimension ful parameters?

$Q_0$, $(d, z)$, $(\rho, R)$ $(\omega, c)$

What are the dimensionless parameters?

$\frac{\omega R}{c} \ll 1$ and $\frac{d}{R}, \frac{z}{R} \ll 1$ and $\frac{\rho}{R}$

So

$$E = \frac{Q}{R^2} f_E\left(\frac{\omega R}{c}, \frac{\rho}{R}\right) + \cancel{\frac{z}{R} g_E} + \cancel{\left(\frac{z^2}{R}\right) h_E} \cdots$$

small

$$B = \frac{Q}{R^2} f_B\left(\frac{\omega R}{c}, \frac{\rho}{R}\right) + O(z/R)$$

Capacitor pg. 2 (dimensional analysis)

So since $\omega R/c \ll 1$

$$E = \frac{Q}{R^2}\left[ f_E^{(0)}\left(\frac{\rho}{R}\right) + \cancelto{0}{\left(\frac{\omega R}{c}\right) f_E^{(1)}}\left(\frac{\rho}{R}\right) + \left(\frac{\omega R}{c}\right)^2 f_E^{(2)} + \cdots \right]$$

Sim

E is T-even
but $\omega$ is T-odd

$$B = \frac{Q}{R^2}\left[ \cancelto{0}{f_B^{(0)}\left(\frac{\rho}{R}\right)} + \left(\frac{\omega R}{c}\right) f_B^{(1)} + \cancelto{0}{\left(\frac{\omega R}{c}\right)^2 f^{(2)}} + \cdots \right]$$

B is time reversal odd, but these are even

Summary

$$E = \frac{Q}{R^2}\left[ f_E^{(0)}\left(\frac{\rho}{R}\right) + \left(\frac{\omega R}{c}\right)^2 f_E^{(2)}\left(\frac{\rho}{R}\right) + \cdots \right]$$

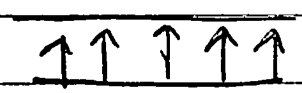
$$B = \frac{Q}{R^2}\left[ \left(\frac{\omega R}{c}\right) f_B^{(1)}\left(\frac{\rho}{R}\right) + \left(\frac{\omega R}{c}\right)^3 f_B^{(3)} + \cdots \right]$$

Capacitor pg. 3

Ok How do we solve:

<u>0th</u> $\nabla \cdot E^{(0)} = 0$

$\nabla \times E^{(0)} = 0$



$E^{(0)} = \frac{Q_0}{\pi R^2} \cos\omega t \; \hat{z}$

<u>1st</u> The $\nabla \times B^{(1)} = \frac{1}{c} \partial_t E^{(0)}$

These follow from

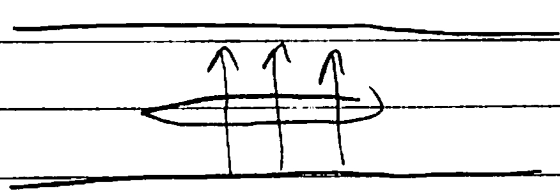
<u>2nd</u> $-\nabla \times E^{(2)} = \frac{1}{c} \partial_t B^{(1)}$

$\nabla \times B = \frac{1}{c} \partial_t E$

$-\nabla \times E = \frac{1}{c} \partial_t B$

<u>1st Order</u>

The displacement current $\equiv \partial_t E^{(0)}$ sources B :



$\nabla \times B^{(1)} = \frac{1}{c} \partial_t E^{(0)}$

$$\underbrace{\oint \vec{B} \cdot d\vec{l} = \frac{1}{c} \int \partial_t E^{(0)} \, 2\pi \rho \, d\rho}_{\text{or solve}}$$

$\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\phi^{(1)}) = \frac{1}{c} \partial_t E^{(0)}$

with

$E^{(0)} = \frac{Q_0}{\pi R^2} \cos\omega t$

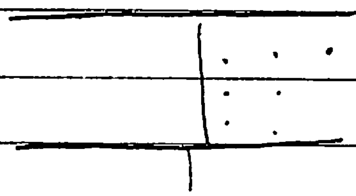
# Capacitor pg. 4

Find

$$B^{(1)}_\phi = \frac{Q_0 \sin\omega t}{\pi R^2}\left(\frac{\omega\rho}{2c}\right)\hat{\phi} \ll E^{(0)}_\phi$$

## 2nd Order.

$$-\nabla\times E^{(2)} = \frac{1}{c}\partial_t B^{(1)}_\phi\,\hat{\phi}$$



Using the expression $\nabla\times E = -\partial E^z/\partial\rho\,\hat{\phi}$

$$+\frac{\partial E^{(2)}_z}{\partial\rho}\,\hat{\phi} = +\frac{1}{c}\partial_t B^{(1)}$$

So plugging $B^{(1)} \propto \rho$ and integrating $\int d\rho$ we find:

$$E^{(2)} = -\frac{Q_0\cos\omega t}{\pi R^2}\,\frac{\omega^2\rho^2}{4c^2}\,\hat{z}$$

Can continue in this way,

$$B^{(3)}_\phi = -\frac{Q_0\sin\omega t}{\pi R^2}\left(\frac{-1}{16}\left(\frac{\omega\rho}{c}\right)^3\right)$$

So

$$\boxed{\begin{aligned} E_z &= \frac{Q_0 \cos\omega t}{\pi R^2}\left[1 - \frac{1}{4}\left(\frac{\omega\rho}{c}\right)^2 + \cdots\right] \\ B_\phi &= \frac{-Q_0 \sin\omega t}{\pi R^2}\left[\frac{\omega\rho}{2c} - \mathcal{O}\left(\frac{\omega R}{c}\right)^3 + \cdots\right] \end{aligned}}$$

Finally we compute the time averaged electric and magnetic energies

$$\langle \cos^2\omega t\rangle = \frac{1}{2}$$

$$\langle \sin^2\omega t\rangle = \frac{1}{2}$$

So

$$U_E = \left\langle \int \frac{1}{2}\left(E^{(0)} + \delta E^{(2)}\right)^2 dV \right\rangle = \left\langle \int dV \, \frac{1}{2}E_0^2 + E_0\,\delta E^{(2)} \right\rangle$$

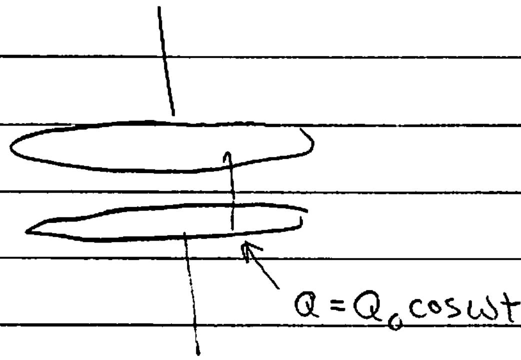
$$U_E \simeq \frac{Q_0^2 d}{4\pi R^2}\left[1 - \frac{1}{8}\left(\frac{\omega R}{c}\right)^2 + \cdots\right]$$

$$U_B = \left\langle \int dV \, \frac{1}{2} B^2 \right\rangle$$

$$= \frac{Q_0^2 d}{4\pi R^2}\left[\frac{1}{8}\left(\frac{\omega R}{c}\right)^2\right]$$

Thus we see that to order $\omega^2$ the energy is shifted

# An important example (2nd Time) in Coulomb Gauge




$$-\nabla^2 \varphi = \rho$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{A} = \frac{\vec{j}}{c} + \frac{1}{c}\partial_t(-\nabla\varphi)$$

**0th** : Then the zeroth solution in $1/c$ :

$$-\nabla^2\varphi = \cancel{\rho}^{\,0} \qquad \vec{A} = 0$$

Find

$$\varphi = -\frac{Q(t)}{\pi R^2}\, z \Leftarrow \text{actually true to all orders}$$

**1st** : At first order ;

$$\left(\underbrace{\cancel{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}_{\text{2nd}} - \nabla^2\right)\vec{A} = \cancel{\frac{\vec{j}}{c}}^{\,0} + \overbrace{\frac{1}{c}\partial_t(-\nabla\varphi)}^{\text{Source}}$$

$$-\nabla^2\vec{A} = -\frac{Q_0 \sin\omega t}{\pi R^2}\left(\frac{\omega}{c}\right)\hat{z}$$

So

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}A^z = -\frac{Q_0\sin\omega t}{\pi R^2}\left(\frac{\omega}{c}\right)$$

So integrating

$$A^z = -\frac{Q_0 \sin\omega t}{\pi R^2}\left(\frac{-\omega\rho^2}{4c}\right) + \underbrace{\text{fcn of } z}$$

But the gauge condition $\nabla\cdot A$

$$\cancel{\partial_x A^x} + \cancel{\partial_y A^y} + \partial_z A^z = 0$$

fixes that fcn of $z$ = at most constant,

Then

$$\vec{B} = \nabla\times\vec{A}$$

$$B_\phi = -\frac{\partial A^z}{\partial\rho} \leftarrow$$ note that $B_\phi$ is indep of fcn of $z$ any way

$$B_\phi = -\frac{Q_0}{\pi R^2}\sin\omega t\left(\frac{\omega\rho}{2c}\right)$$

Agrees w before

2nd: Note that the second order E-field is very easy to work out in the coulomb gauge

$$\varphi = -\frac{Q}{\pi R^2}\cos\omega t\, z \quad \text{is exact}$$

Further $\vec{A}(t,z)$ is reversal odd so $\vec{A}(t,z)$ must be odd in frequency, so can not have second order terms

$$\text{So} \quad E = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$$

$$\vec{E} = \underbrace{-\frac{1}{c}\partial_t \vec{A}^{(1)}}_{\vec{E}^{(2)}} + \vec{E}^{(0)}$$

← using $A^{(1)}$ from previous page

Thus

$$\boxed{E^{(2)} = Q_0 \frac{\cos\omega t}{\pi R^2}\left[-\left(\frac{\omega\rho}{c}\right)^2 \frac{1}{4}\right]}$$

Same as before