An important example (2nd Time) in Coulomb Gauge

\[- \nabla^2 \phi = \rho\]

\[\alpha = Q_0 \cos \omega t\]

\[\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \hat{A} = \frac{j_0}{c} + \frac{1}{c} \int \rho (-\nabla \phi) \]

\[\text{Oth: Then the zeroth solution in } V_0:\]

\[- \nabla^2 \phi = \rho \]

\[\hat{A} = 0\]

Find

\[\phi = -\frac{Q(t)}{\pi R^2} \]

\[\approx \text{ actually true to all orders}\]

1st: At first order:

\[\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \hat{A} = \frac{j_0}{c} + \frac{1}{c} \int \rho (-\nabla \phi) \]

\[- \nabla^2 \hat{A} = -Q_0 \sin \omega t \frac{w}{\pi R^2} \]

So

\[- \frac{1}{\rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \frac{A^2}{\rho} = -Q_0 \sin \omega t \frac{w}{\pi R^2} \]

So integrating

\[ A^2 = -\frac{Q}{\pi R^2} \sin\omega t \left( -\frac{\omega p^2}{4c} \right) + \text{fun of } z \]

But the gauge condition \( \nabla \cdot A \)

\[ 2_x A_x + 2_y A_y + 2_z A_z = 0 \]

fixes that fun of \( z \) = at most constant,

Then

\[ \vec{B} = \nabla \times \vec{A} \]

\[ B_\phi = -\frac{\partial A^2}{\partial \phi} \text{ note that } B_\phi \text{ is } \]

\[ \text{indep of fun of } z \text{ anyway} \]

\[ \frac{B_\phi}{\pi R^2} = \frac{-Q}{\pi R^2} \sin\omega t \left( \frac{\omega p}{2c} \right) \]

Agrees \( \odot \) before

2nd: Note that the second order E-field is

very easy to work out in the coulomb
gauge

\[ \Phi = -\frac{Q}{\pi R^2} \cos\omega t \text{ is exact} \]
Further $\hat{A}(t, z)$ is reversal odd so $\hat{A}(t, z)$ must be odd in frequency, so cannot have second order terms.

So

$$E = -\frac{1}{c} 2\hat{A} - \nabla \Phi$$

$$\vec{E} = -\frac{1}{c} 2\hat{A}^{(1)} + \hat{E}^{(0)}$$

Thus

$$\vec{E}^{(2)} = \text{using } A^{(1)} \text{ from previous}$$

$$\vec{E}^{(2)} = \frac{Q_0 \cos \omega t}{4\pi R^2} \left[ \begin{array}{c} -\left(\frac{\omega \rho}{c}\right)^2 \frac{1}{c} \\ \left(\frac{\omega \rho}{c}\right)^2 \frac{1}{4} \end{array} \right]$$

Same as before
Quasi-statics + Induction in metals

- diffusion of magnetic fields
  metal

\[ H = H_0 e^{-i\omega t} \]

\[ H_0 \text{ decreasing} \]

We will find that induced currents cause the magnetic fields to decay in metal.

1. If the magnetic fields are increasing (as drawn) which way do the currents flow?

2. What are the dimensionful parameters?

\[ H_0, \omega, c, \sigma \]

Then

\[
\left[ \sigma \right] = \frac{1}{S} \quad \sigma \approx 10^8 \text{ Hz for Cu}
\]

We will see that a characteristic scale for decay is

\[
S = \sqrt{\frac{2c^2}{\sigma \omega}} = \left( \frac{m/s}{\text{v}_s \text{v}_s} \right)^2 \sim m
\]
Analysis of Quasi-statics in metals

\[ \nabla \cdot E = 0 \]

\[ \nabla \times H = j_{\text{ind}} \]

\[ \nabla \cdot B = 0 \]

\[ -\nabla \times E = \frac{1}{c} \frac{\partial B}{\partial t} \]

So \( j_{\text{ind}} = \sigma E_{\text{ind}} \) then we have with \( B = \mu H \)

\[ \nabla \times H = \frac{\sigma E_{\text{ind}}}{c} \]

\[ \nabla \cdot \nabla \times H = \frac{\sigma}{c} \nabla \times E_{\text{ind}} \]

\[ \nabla \times E = \frac{\sigma}{c} \frac{\partial H}{\partial t} \]

\[ \nabla (\nabla \cdot H) - \nabla^2 H = -\frac{\sigma \mu}{c^2} \frac{\partial H}{\partial t} \]

So find a diffusion equation for magnetic fields:

\[ \nabla^2 H = \frac{\sigma \mu}{c^2} \frac{\partial H}{\partial t} \]

Diffusion equation

\[ \frac{\partial t}{\partial t} n = D \nabla^2 n \]

canonical form
diffusion coefficient
A primer on Diffusion Equation:

A drop of dye in water

\[ t=0 \]

The width of the drop increases in time

\[ (\Delta x)^2 = 2Dt \]

\[ \uparrow \]

diffusion coefficient

- The diffusion equation smears out features

- The magnetic diffusion coefficient:

\[ D = \frac{C^2}{\mu \sigma} \]

\[ \mu \text{ is dimensionless} \]

\[ \approx \frac{1 \text{ cm}^2}{\text{millisecond}} \] for Cu \( \mu = 1 \) \( \sigma = 10^{18} \text{ Hz} \)

Solving the diffusion equation

\[ \hat{H}(x, t) = H_0 e^{iwt} h(x) \]

Then substitute into

\[ \nabla^2 H = \frac{1}{D} \partial_t H \]
Solving the Diff Eq. pg. 2
Then find $\partial_t H \propto -i\omega H$

$$\left( \frac{\partial^2}{\partial x^2} + \frac{i\omega}{D} \right) h(x) = 0$$

So try $h(x) = c e^{ikx}$:

$$-k^2 + \frac{i\omega}{D} = 0 \implies k_{\pm} = \pm (1 + i) \sqrt{\frac{\omega}{2D}}$$

Note $t\sqrt{i} = t(1 + i) \sqrt{\frac{1}{2}}$

Thus $e^{ik_+x} = e^{ix/s} e^{-x/s}$

while $e^{-ik_-x} = e^{-ix/s} e^{x/s} \quad \text{Discard}$

So

$$H(x,t) = \text{Re} \left( H_0 e^{i\omega t} e^{ix/s} - x/s \right)$$

$$H(x,t) = H_0 e^{-x/s} \cos \left( \frac{x}{s} - \omega t \right)$$

$H_0 \uparrow$

$X/s$
Diff. Eq. pg. 3

Thus find that the magnetic field decays with characteristic length, $S$

$$S = \sqrt{\frac{2D}{\nu}} = \sqrt{\frac{2c^2}{\omega \mu_0 \sigma}} \quad \sigma \sim 10^{18} \text{ Hz}$$

For $D_{cu} \sim \frac{cm^2}{\text{millisecond}}$, find $S \sim \frac{cm}{\sqrt{f_{kHz}}}$

(property of metal)

(property of metal and probe)

We can calculate the electric field

$$\frac{d^2 x}{dx} = \sigma E = \nabla \times B$$

Find for $B$ in $z$-direction

$$\frac{d^2 y}{dx} = -\alpha B^2 = \text{Re} \left[ -2 H_0 e^{-i\omega t} e^{i k_x x} \right]$$

$$= \text{Re} \left[ -ik \cdot H_0 e^{-i\omega t} e^{i k_x x} \right]$$

$$\frac{d^2 y}{dx} = \sqrt{2} \frac{H_0}{c} e^{-x/\delta} \cos \left( \frac{x}{\delta} - \omega t - \frac{T}{4} \right)$$
Analysis of Diffusion Eq. pg. 1

1. So a parametric estimate for $E_{\text{ind}}$ is:

$$E_{\text{ind}} \sim \frac{j/c}{\sigma/c} \sim \frac{cH_0}{\sigma} \quad S = \frac{2e^2}{\sqrt{\omega \mu_0}}$$

$\mu \sim 1$

$$E_{\text{ind}} \sim \sqrt{\frac{\omega}{\sigma}} H_0$$

For $E_{\text{ind}}$ to be small (which we assumed), we must have

$$\omega \ll \sigma \quad \sigma \sim 10^{18} \text{ Hz}$$

which is satisfied deep into optical frequencies.

2. Let's compute the total current.

If I look at this from far away, what do I see?
Analysis of Diffusion Eq. pg. 2

\[ H = 0 \quad \text{I see this} \]

(I can't see the boundary layer of width \( \sim \delta \))

\[ K^y = \int_{-\delta}^{\delta} \sqrt{2} H_0 e^{-x/\delta} \cos \left( \frac{x - \omega t - \pi/4}{\delta} \right) \, dx \]

\[ \frac{K^y}{c} = H_0 \cos \omega t \]

This is what you expect from boundary conditions.

\[ n \times \left( \frac{\partial}{\partial z} H_{out} - \frac{\partial}{\partial z} H_{in} \right) = \frac{K}{c} \]

\[ + n \times \frac{\partial}{\partial z} H_{out} = \frac{K}{c} \]

\[ H_{out} = H_0 \cos \omega t \hat{z} \]

\[ H_0 \cos \omega t = \frac{K^y}{c} \]