Last Time

	Lorentz	Coulomb
① $\nabla \cdot E = \rho$	$-\Box \varphi = \rho$	$-\nabla^2 \varphi = \rho$
② $\nabla \times B = \vec{j}/c + \frac{1}{c}\partial_t \vec{E}$	$-\Box A = J/c$	$-\Box A = \frac{\vec{j}}{c} + \frac{1}{c}\partial_t(-\nabla\varphi)$

$$\nabla \times B = 0 \qquad \vec{B} = \nabla \times \vec{A}$$

$$-\nabla \times E = \frac{1}{c}\partial_t \vec{B} \qquad \vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$$

Then

Coulomb $\nabla \cdot \vec{A} = 0$

① $-\nabla^2 \varphi = \rho$

② $-\Box A = J/c + \frac{1}{c}\partial_t(-\nabla\varphi)$

Lorentz

$\frac{1}{c}\partial_t \varphi + \nabla \cdot \vec{A} = 0$

① $-\Box \varphi = \rho$

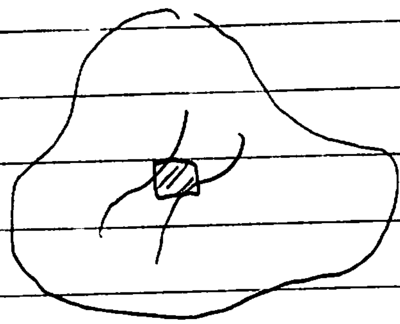
② $-\Box \vec{A} = \vec{J}/c$

Also Discussed Quasi-Statics in metals

$$\nabla \times B = \underbrace{\sigma E^{ind}/c}_{j/c} + \underbrace{\cancel{\frac{1}{c}\partial_t E^{ind}}}_{\text{small}}$$

$$+\nabla^2 B = \frac{1}{D}\partial_t B \qquad D = \frac{c^2}{\mu\sigma} \equiv \text{magnetic diffusion}$$

Energy Conservation for Linear Matter:



The energy density

- u_{mech} = mechanical energy / vol (compression, thermal)
- u_{em} = The electric and magnetic energy/vol
- u_{TOT} = total $= u_{mech} + \underbrace{u_{em}}_{\frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{B}\cdot\vec{H}}$

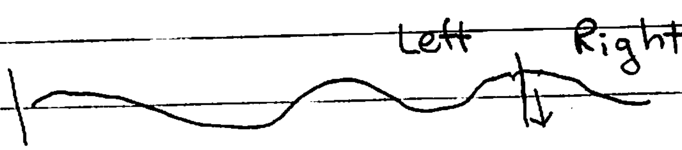
- For the total energy flux Expect:

$$\partial_t u_{TOT} + \partial_i S^i_{TOT} = 0$$

In this way an isolated system will conserve energy.

- The energy flux $\vec{S}_{TOT}$ has mechanical pieces, $\vec{S}_{mech}$, and electromagnetic flux

Ex: Stretched string



$$u_{mech} = \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2$$

$$\vec{S} = \underbrace{T_0 \frac{\partial y}{\partial x}}\,\underbrace{\frac{\partial y}{\partial t}} = \underbrace{\text{Force}} \cdot \underbrace{\text{velocity}}$$

S_{mech} records how energy is mechanically transported from one region to another

We will show:

$$\partial_t (u_{mech} + u_{em}) + \partial_i (S^i_{mech} + S^i_{em}) = 0$$

Where $u_{em} = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{B}\cdot\vec{H}$

and $\vec{S}_{em} = c(\vec{E}\times\vec{H})$

energy flux in electromagnetism

In integral form:

$$\frac{d(U_{mech} + U_{em})}{dt} = -\int \vec{S}_{em}\cdot d\vec{a} - \underbrace{\cancel{\int \vec{S}_{mech}\cdot d\vec{a}}}_{0}$$

0 for a mechanically isolated system

Then

$$[S] = \frac{\text{energy}}{\text{vol}}\cdot\frac{m}{s} = \frac{\text{energy}}{\text{area } s}$$

Prf - Imagine a charged string

$$\boxed{\partial_t u_{mech} + \partial_i S^i_{mech} = \vec{j} \cdot \vec{E}}$$

Now

$$\vec{j} \cdot \vec{E} = c\left(\nabla \times H - \frac{1}{c}\partial_t \vec{D}\right) \cdot \vec{E}$$

$$= \underbrace{c\,(\nabla \times H) \cdot E} - E\,\partial_t \vec{D}$$

equivalent

$$= \nabla \cdot (c\,H \times E) + H\,c\,\underbrace{\nabla \times E}_{= -\frac{1}{c}\partial_t \vec{B}}$$

$$= -\nabla \cdot S_{em} - (E\,\partial_t D + H\,\partial_t B)$$

$$= -\nabla \cdot S_{em} - \partial_t u_{em}$$

And so

$$\boxed{\partial_t (u_{mech}) + \partial_i S^i = -\partial_t u_{em} - \partial_i S^i_{em}}$$

$$u_{em} = \frac{1}{2} E \cdot D + \frac{1}{2} B \cdot H$$

$$\vec{S}_{em} = c\,\vec{E} \times \vec{H}$$

Momentum Conservation

$$\partial_t g^j_{TOT} + \partial_i T^{ij}_{TOT} = 0$$

- g_{TOT} is the total momentum per volume
- T^{ij}_{TOT} is the force in the i-th derecton per area in j-th
- This guarantees that the total momentum is conserved

$$\frac{dP^j_{TOT}}{dt} = \int dV\, \partial_t g^j_{TOT} = \int dV \left[-\partial_i T^{ij}\right]$$

$$= -\int_{\partial V} T^{ij} n_j \, dS = 0$$

for an isolated system

Will show that

$$\partial_t g^j_{mech} + \partial_i T^{ij}_{mech} = \rho \vec{E} + \frac{\vec{j}}{c} \times \vec{B}$$

$$= -\partial_t (\vec{g}^{\,j}_{em}) - \partial_i T^{ij}_{em}$$

same

Where

$$\vec{g}_{em} = \frac{\varepsilon\mu}{c^2} \vec{S}_{em} \qquad T^{ij}_{em} = \overbrace{-\varepsilon E^i E^j + \frac{1}{2}\varepsilon E^2 \delta_{ij}}^{\text{electric stress}}$$

$$+ \; -\frac{B^i B^j}{\mu} + \frac{1}{2}\frac{B^2}{\mu}\delta_{ij}$$ ← magnetic stress

So that the full result:

$$\partial_t(\vec{g}_{mech} + \vec{g}_{em}) + \partial_i(T^{ij}_{mech} + T^{ij}_{em}) = 0$$

<u>Prf</u>

$$\partial_t g^j_{mech} + \frac{\partial T^{ij}_{mech}}{\partial x^i} = f^j_{em}$$

Now write

$$f^j_{em} = \rho E^j + \left(\frac{\vec{j}}{c} \times \vec{B}\right)^j$$

Then use $\nabla \cdot D = \rho$ $\quad \frac{\vec{j}}{c} = \nabla \times \vec{H} - \frac{1}{c}\partial_t \vec{D}$

Find

$$f^j_{em} = \underbrace{(\nabla \cdot D) E^j}_{1} + \underbrace{\left[(\nabla \times \vec{H}) \times \vec{B}\right]^j}_{2} - \underbrace{\frac{1}{c}(\partial_t \vec{D} \times \vec{B})^j}_{3}$$

The rest is labor wich I will not go through (see Jackson)

(1) $\Rightarrow$ $-\partial_i T^{ij}_E$ with $T^{ij}_E = -\varepsilon E^i E^j + \frac{1}{2}\varepsilon E^2 \delta^{ij}$

(2) $\Rightarrow$ $-\partial_i T^{ij}_B$ with $T^{ij}_B = -\frac{B^i B^j}{\mu} + \frac{1}{2}\frac{B^2}{\mu}\delta^{ij}$

(3) $\Rightarrow$ $-\partial_t \vec{g}_{em}$ with $\vec{g}_{em} = \vec{D}\times\vec{B} = \frac{\varepsilon\mu}{c^2}\vec{S}$

(plus a term from (1))

The result is: $\boxed{\rho \vec{E}^j + J/c \times \vec{B} = -\partial_t g^j_{em} - \partial_i T^{ij}_{em}}$

$$\partial_t(g^j_{mech} + g^j_{em}) + \partial_i(T^{ij}_{mech} + T^{ij}_{em}) = 0$$

same thing

$$\partial_t g^j_{mech} + \partial_i T^{ij}_{mech} = \rho E^j + \left(\frac{\vec{J}}{c}\times\vec{B}\right)^j$$

Or in terms of integralts:

$$\boxed{\frac{d(P^j_{mech} + P^j_{em})}{dt} = -\int da\, n_i T^{ij}_{em}} - \cancel{\int da\, n_i T^{ij}_{mech}}$$

Total momentum

0 for a mechanically isolated syste

$$\vec{P} = \int dV\, \vec{g}$$