Plane Waves in Linear Matter

(1) $\nabla \cdot D = 0$

(2) $\nabla \times H = \frac{1}{c} \partial_t D$

(3) $\nabla \cdot B = 0$

(4) $-\nabla \times E = \frac{1}{c} \partial_t B$

Note a symmetry in absence of currents

$H \to -E$ and $D \to B$

In vacuum: $B \to -E$ and $E \to B$ (Electric - magnetic duality)

Then to derive the wave-eqn in linear matter we take curl of (2) and use "bac - abc"

$$\nabla \times (\nabla \times H) = \frac{1}{c} \partial_t (\nabla \times \vec{D})$$

$$\nabla (\cancel{\nabla \cdot \vec{H}}) - \nabla^2 \vec{H} = -\frac{\mu \varepsilon}{c^2} \partial_t^2 H$$

$D = \varepsilon E$
$B = \mu H$

$$\boxed{\left(\frac{\mu \varepsilon}{c^2} \partial_t^2 - \nabla^2 \right) H = 0}$$

By symmetry

$$\boxed{\left(\frac{\mu \varepsilon}{c^2} \partial_t^2 - \nabla^2 \right) E = 0}$$

Now we pass from the time dependent wave-eqn to the time-indepent wave eqn

$$H(x,t) = e^{-i\omega t} H(x) \qquad E(x,t) = e^{-i\omega t} E(x)$$

To find the Helmholtz equations:

$$\boxed{\begin{gathered}\left[\omega^2\left(\frac{\mu\varepsilon}{c^2}\right) + \nabla^2\right]\vec{H}(x) = 0 \\[1em] \left[\omega^2\left(\frac{\mu\varepsilon}{c^2}\right) + \nabla^2\right]\vec{E}(x) = 0\end{gathered}}$$

This is an equation for the allowed eigen frequencies and the corresponding modes. The general solution is a super-position of these modes

Try

$$E = \vec{\mathcal{E}}\, e^{i\vec{k}\vec{x} - i\omega t}$$

$$\vec{H} = \vec{\mathcal{H}}\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

Find

$$-k^2 + \omega^2\left(\frac{\mu\varepsilon}{c^2}\right) = 0 \quad \Rightarrow \quad \omega = \frac{c\,k}{n}$$

where

$n = \sqrt{\mu\varepsilon}$ is the index of refraction

Propeties of Plane Waves

The phase velocity of the wave is

$$V_\phi = \frac{\omega}{k} = \frac{c}{n}$$

Now the divergence Eqs

$$\nabla \cdot E = 0$$

$$\nabla \cdot B = 0$$

give rise to

$$\left.\begin{array}{l} \vec{k} \cdot \vec{\mathcal{E}} = 0 \\ k \cdot \mathcal{H} = 0 \end{array}\right\}$$

Thus the vectors $\vec{\mathcal{E}}$ & $\vec{\mathcal{H}}$ are transvere to the beam

Finally we have

$$\nabla \times E = -\frac{1}{c} \partial_t B$$

$$i \vec{k} \times \vec{\mathcal{E}} = \frac{i\omega}{c} \mu \vec{\mathcal{H}}$$

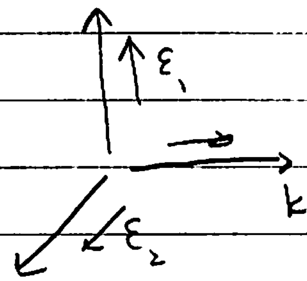
For $k = \omega/(c/n)$ find using $\hat{k} = \vec{k}/k$

$$\frac{1}{Z} \vec{k} \times \vec{\mathcal{E}} = \vec{\mathcal{H}} \qquad \text{with} \quad Z = \sqrt{\frac{\mu}{\varepsilon_0}}$$

$\vec{\mathcal{H}}$ is $\frac{1}{Z}$ relative to $\vec{\mathcal{E}}$, or $\mathcal{B}$ is $\sqrt{\mu\varepsilon}$ relative $\vec{\mathcal{E}}$.

Summary of Polarization

- Construct two vectors which are orthogonal to $\vec{k}$



Found

$$\vec{\mathcal{E}} = \vec{\varepsilon}_1 E_0 \qquad \vec{\mathcal{H}} = \frac{1}{Z}\hat{k} \times \mathcal{E}_0 = \frac{1}{Z}\vec{\varepsilon}_2 E_0$$

or the reverse: $\mathcal{E} \to \vec{H} \qquad \mathcal{H} \to -\mathcal{E}$

$$\vec{\mathcal{E}} = -\vec{\varepsilon}_2 E_0 \qquad \mathcal{H} = \frac{1}{Z}(+\vec{\varepsilon}_1) E_0$$

In general we take a complex superposition

$$\vec{E} = (\vec{\varepsilon}_1 E_1 + \vec{\varepsilon}_2 E_2) e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

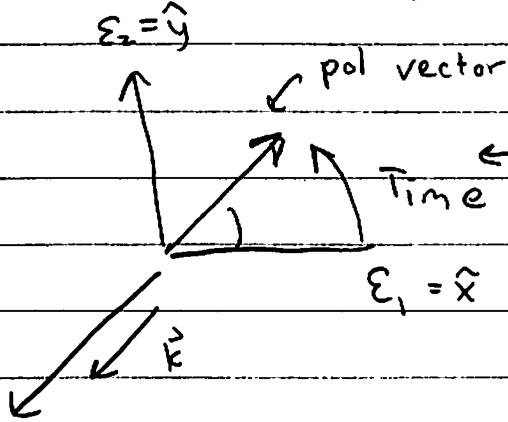
(1) If E_1 and E_2 are in phase, then the light is linearly polarized

(2) If E_1 and E_2 are out phase by 90° (and equal in magnitude) the light is circularly polarized

$$\vec{E} = E_0 (\vec{\varepsilon}_1 \pm i \vec{\varepsilon}_2) e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

So

$$\mathrm{Re}\,\vec{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_0 \cos(kz - \omega t) \\ \mp \sin(kz - \omega t) \end{pmatrix}$$



So from the picture we have,

$\varepsilon_1 + i\varepsilon_2$ has **positive** helicity

while

$\varepsilon_1 - i\varepsilon_2$ has **negative** helicity

Time Averaging

Now Consider the time averaged poynting vector

$$\langle \vec{S} \rangle = c \langle \mathrm{Re}\, \vec{E} e^{-i\omega t} \times \mathrm{Re}\, H e^{-i\omega t} \rangle$$

$$= c \left\langle \left(\frac{\vec{E} e^{-i\omega t} + \vec{E}^* e^{i\omega t}}{2} \right) \times \left(\frac{\vec{H} e^{-i\omega t} + H^* e^{i\omega t}}{2} \right) \right\rangle$$

$\sim e^{-2i\omega t}$

$$= \frac{1}{4} c \left(\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H} \right) + \langle \text{oscillating} \rangle$$

$$= \frac{1}{2} c\, \mathrm{Re} \left(E \times H^* \right)$$ ← In general take half the real part

Similarly we have:

Re unnecessary since this is real

$$u = \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \frac{B^2}{\mu} \Rightarrow \langle u \rangle = \frac{1}{4} \left(\varepsilon E \cdot E^* + \mu H \cdot H^* \right)$$

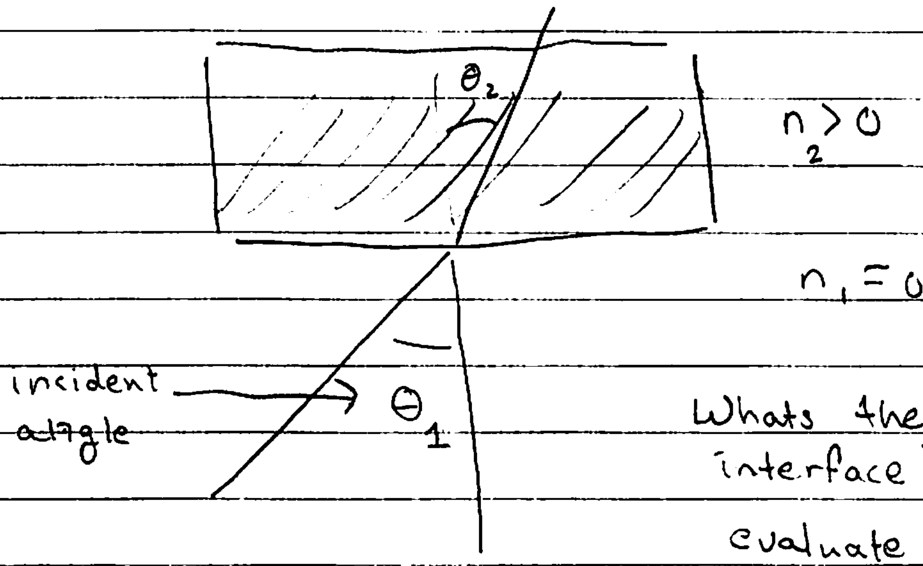
And

$$\langle T^{ij}_E \rangle = \frac{\mathrm{Re}}{2} \left\langle -E^i E^{j*} + \frac{1}{2} E \cdot E^* \delta_{ij} \right\rangle$$

$$= \frac{1}{2} \left(E^i E^{j*} + E^{i*} E^j + \frac{1}{2} E \cdot E^* \delta_{ij} \right)$$

Reflection of Light at Interfaces

• Qualitative:



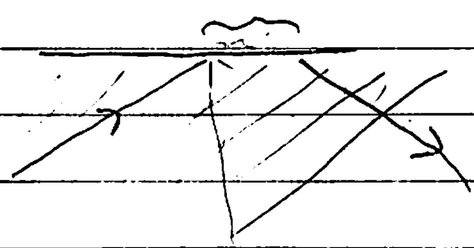
Whats the force on the interface? Solve for fields evaluate stress

• Snells Law $\quad n_1 \sin\theta_1 = n_2 \sin\theta_2 \Leftarrow$ want to derive

• Internal Reflection

shifted ~ Goos - Hänchen effect

tunnelling

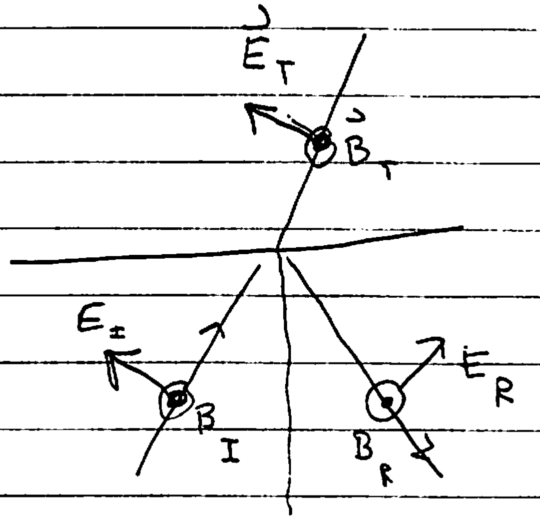


• This happens when $n_1 > n_2$, $\sin\theta_2 > \sin\theta_1$

• Evaluate forces using stress tensor

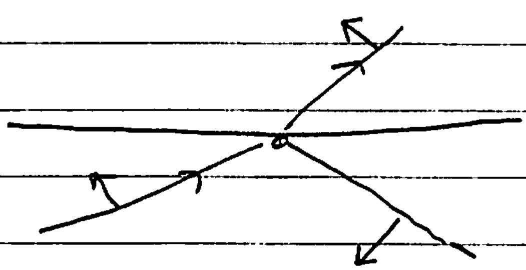
Polarization - Transverse Magnetic (In plane polarization)

(small incident angles
Consider Head on Case;

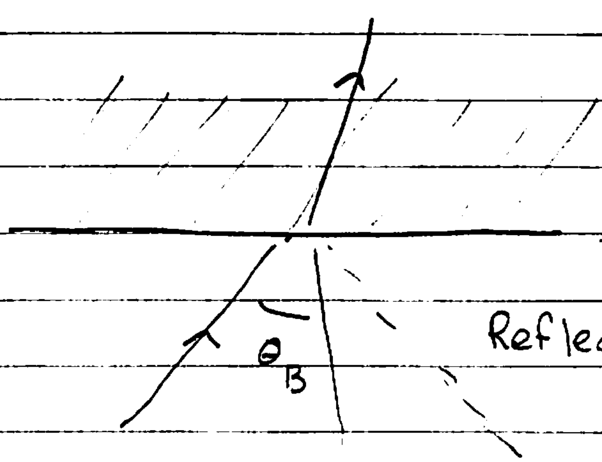


• E inverted on reflection

But for large incident angles $\vec{E}$ not inverted:



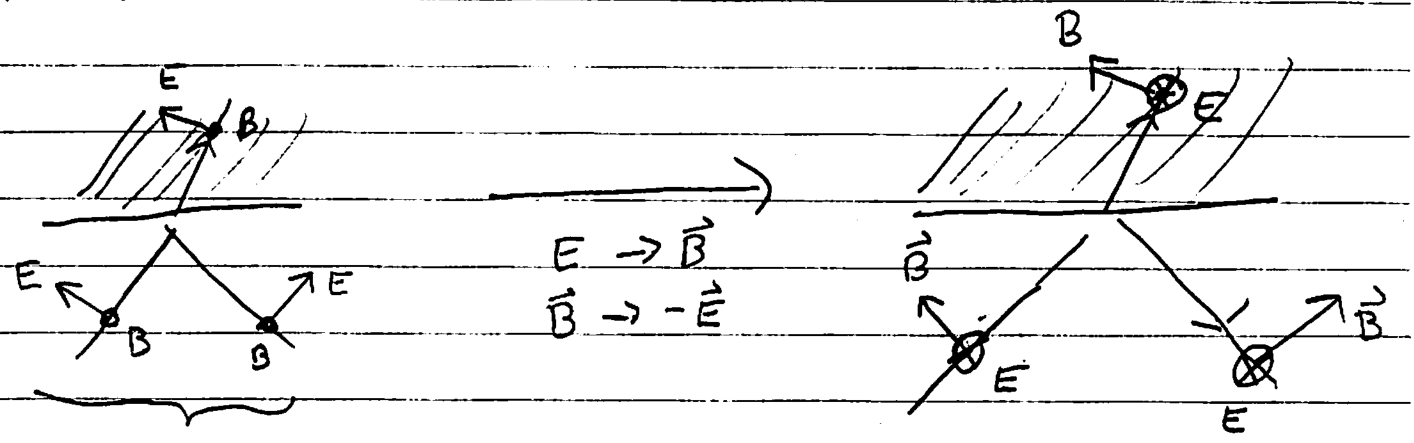
So there must be angle θ_B "Brewster Angle" where there is no reflected wave.



Reflected wave vanishes

$$\tan\theta_B \simeq \frac{n_2}{n_1}$$

In general the incident light is partially polarization:



Transverse Magnetic (in plane)

Transverse Electric (out of plane)

- After going through the interface the reflected / transmitted light will be polarized

  out of plane / in plane

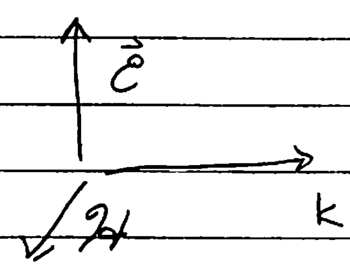
- This is used by radio towers to select transmitted waves

Last Time

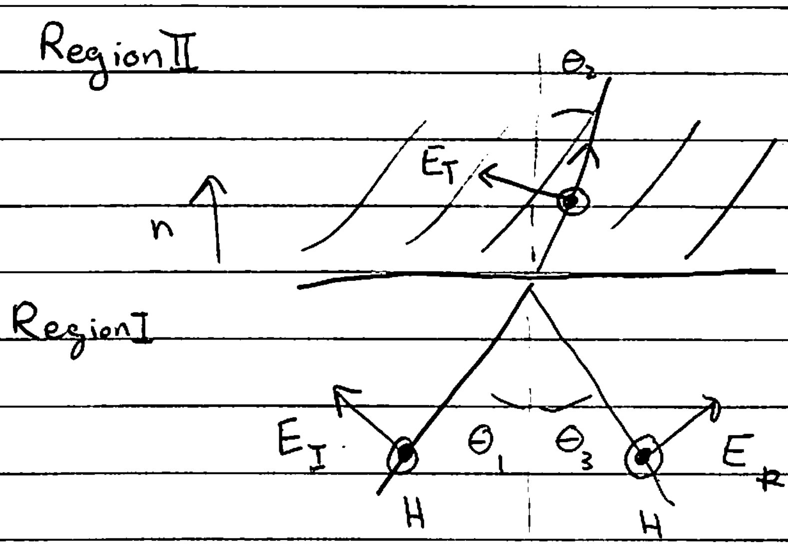
- Discussed the properties of plane waves:

$$\vec{E} = \vec{\mathcal{E}}\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

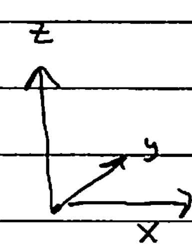
$$\vec{H} = \vec{\mathcal{H}}\, e^{ik\cdot x - i\omega t}, \qquad \vec{\mathcal{H}} = \frac{\hat{k}\times\vec{\mathcal{E}}}{Z} \qquad Z = \sqrt{\frac{\mu}{\epsilon}}$$



- Discussed The essential qualitative features of reflection and transmission at interfaces



$n_2 > n_1$



← call this in plane polarized, or transverse magnetic

Discussed

• Basic idea write the solution in region II and region I as plane waves (sum of), and use boundary conditions to relate the two regions

<u>Region II</u>

~~polarization vector~~

$$\vec{E} = \vec{E}_T \, e^{i\vec{k}_T\cdot\vec{r} - i\omega t}$$

$$\vec{H} = H_T \, e^{i\vec{k}_T\cdot\vec{r} - i\omega t}(-\hat{y}) \leftarrow \text{out of page}$$

transverse magnetic

<u>Region I</u>

$$\vec{E}_1(t,\vec{r}) = \vec{E}_I \, e^{i\vec{k}_I\cdot\vec{r} - i\omega t} + \vec{E}_R \, e^{i\vec{k}_R\cdot\vec{r} - i\omega t}$$

$$\vec{H}_1(t,r) = \left(H_I e^{i\vec{k}_I\cdot\vec{r} - i\omega t} + H_R e^{i\vec{k}_R\cdot\vec{r} - i\omega t}\right)(-\hat{y})$$

<u>Boundary Conditions</u>

$$n\cdot(\vec{D}_2 - \vec{D}_1) = 0$$

$$n\times(\vec{H}_2 - \vec{H}_1) = 0$$ (Parallel components of H continuous)

$$n\cdot(\vec{B}_2 - \vec{B}_1) = 0$$

$$n\times(\vec{E}_2 - \vec{E}_1) = 0$$ (Parallel components of E continuous)

Solving the B.C

- Has to hold at all times and for every point on the interface

$$i\vec{k}_I \cdot \vec{r} - i\omega t \Big|_{z=0} = i\vec{k}_R \cdot \vec{r} - i\omega t \Big|_{z=0} = i\vec{k}_T \cdot \vec{r} - i\omega t \Big|_{z=0}$$

- Frequencies have to be the same, So:

$$k_I = |\vec{k}_I| = |\vec{k}_R| = \frac{\omega n_1}{c}$$

$$|\vec{k}_T| = \frac{\omega n_2}{c}$$

Thus the wavelengths are related $k_T = \frac{n_2}{n_1} k_I$

- At $z=0$, $\vec{k} \cdot \vec{r} \Big|_{z=0} = \vec{k}_x = k \sin\theta$

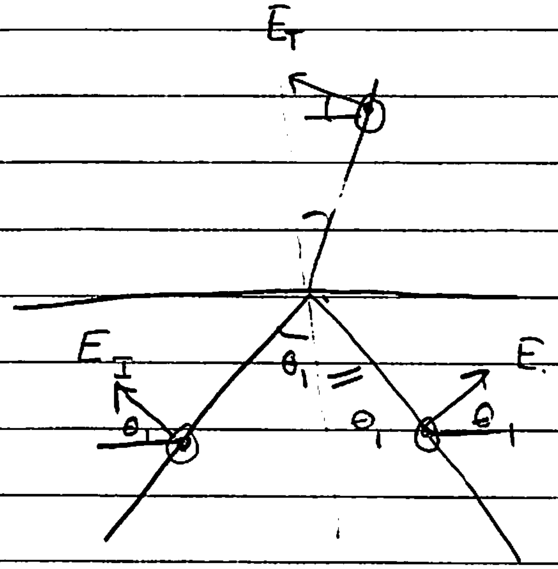
$$k_I \sin\theta_1 = k_R \sin\theta_3 = k_T \sin\theta_2$$

Or $\theta_1 = \theta_3$ incident = reflected

$$\sin\theta_1 = \frac{k_T}{k_I} \sin\theta_2$$

$$\sin\theta_1 = \frac{n_2}{n_1} \sin\theta_2 \quad \text{(snells law)}$$

Now $E_{\parallel}$ continious and $H_{\parallel}$ continuous



So the x-components of E are continuous

$$-E_T \cos\theta_2 - (-E_I \cos\theta_1 + E_R \cos\theta_1) = 0$$

And from continuity of H

$$H_T - (H_I + H_R) = 0$$

So H is related to E, $H = E/Z$

$$+\frac{E_T}{Z_2}\cos\theta_2 - (E_I + E_R)/Z_1 = 0$$

Solving for the reflected + Transmitted amplitudes:

Find

$$\frac{E_R}{E_I} = \frac{Z_1 \cos\theta_1 - Z_2 \cos\theta_2}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \underset{\text{head on}}{\Longrightarrow} \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

$$\frac{E_T}{E_I} = \frac{2 Z_2 \cos\theta_1}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \underset{\text{head on}}{\Longrightarrow} \frac{2 Z_2}{Z_1 + Z_2}$$

Now we want to andlyz this

- Energy Transport

$$\vec{S} = \frac{1}{2} c \, E \times H^* = \frac{c}{2} \frac{1}{Z} |E|^2 \hat{k}$$

$\vec{S}$: time averaged poynting flux

So the transmitted power, relative to the input power

$$T_p = \frac{\vec{S}_T \cdot \vec{n}}{\vec{S}_I \cdot \vec{n}} = \frac{\cos\theta_2}{\cos\theta_1} \frac{Z_1}{Z_2} \frac{|E_T|^2}{|E_I|^2}$$

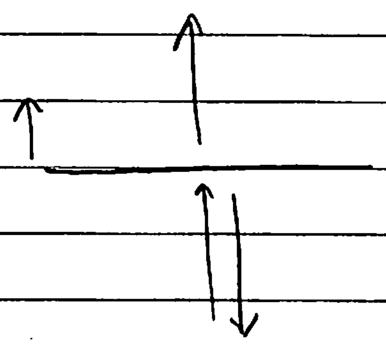
$$\underset{\text{head on}}{\Longrightarrow} \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2}$$

$$R_p = \frac{\vec{S}_R \cdot (-\vec{n})}{\vec{S}_I \cdot \vec{n}} = \frac{\cos\theta_1}{\cos\theta_1} \frac{Z_1}{Z_1} \frac{|E_R|^2}{|E_I|^2} \underset{\text{head on}}{\Longrightarrow} \frac{(Z_1 - Z_2)^2}{(Z_1 + Z_2)^2}$$

Find generally

$$R_p + T_p = 1$$

Momentum Transport:



$n_2 = $ something , $\mu_2 = 1$, $Z_2 = \frac{1}{n_2} \equiv \frac{1}{n}$

$n_1 = 1$, $\mu_1 = 1$, $Z_1 = 1$

$$\frac{\text{Force}}{\text{Area}} = -\left(T^{zz}_{out} - T^{zz}_{in}\right)$$

$H = E/\underset{\sim}{Z}$

$$T^{zz} = \frac{\varepsilon}{2}\left(-\cancel{E^z E^{z*}} + \frac{1}{2} E\cdot E^* \, \delta^{zz}\right) + \frac{\mu}{2}\left(-\cancel{H^z \cdot H^{z*}} + \frac{1}{2}\vec{H}\cdot\vec{H}^* \, \delta^{zz}\right)$$

time ave

$$= \frac{\varepsilon}{4}|E|^2 + \frac{\mu}{4}\frac{1}{\mu/\varepsilon}\,\vec{\underline{E}}\cdot\vec{\underline{E}} = \frac{\varepsilon}{2}\vec{E}\cdot\vec{E}^*$$

So

$$T^{zz}_{out} = \frac{\varepsilon_2}{2}E_T^2 = \frac{\varepsilon_1 E_I^2}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\left|\frac{E_T}{E_1}\right|^2\right)$$

index of refraction

$$= \langle u_I \rangle \, (n T)$$

transmission

incident energy density

Similarly

$$T^{zz}_{in} = \frac{\varepsilon}{2} (E_I + E_R) \cdot (E_I + E_R)^*$$

$$T^{zz}_{in} = \underbrace{\frac{1}{2} \varepsilon_I E_I^2}_{\langle u_I \rangle} (1 + R)$$

Find

$$\frac{\langle \text{Force} \rangle}{\text{Area}} = \langle u_I \rangle \left[1 + R - nT \right]$$

So for $\mu \simeq 1$ and n

$$\langle \text{Force}/\text{Area} \rangle = \langle u_I \rangle \frac{2(n-1)}{n+1} \xrightarrow[n \to \infty]{} 2 \langle u_I \rangle$$

Total reflection

Note:

$$\langle u_I \rangle = c \langle g_{em} \rangle = \langle S \rangle / c$$

energy density $\simeq \frac{m}{s}$ $\frac{\text{momentum}}{\text{Vol}}$ $\frac{\text{Energy}}{\text{Area} \cdot \text{time}} \cdot \frac{1}{m/s}$