Vectors - Summation Convention:

- Notation:

$$\vec{V} = V^1 \vec{e}_1 + V^2 \vec{e}_2 + V^3 \vec{e}_3 = \sum_{i=1}^{3} V^i \vec{e}_i$$

Or

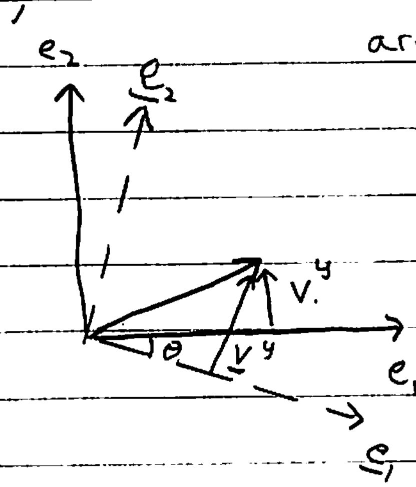
$$\vec{V} = V^x \hat{i} + V^y \hat{j} + V^z \hat{k}$$

Then use summation convention = repeated indices are summed

$$\vec{V} = V^i \vec{e}_i$$

- Physical Objects:

Vectors are physical objects. If coordinates are rotated, the vector is unchanged. But coordinates are changed



$$\underline{V}^i = R^i_{\ j} V^j$$

Think of V^i as a column:

$$\begin{pmatrix} V^x \\ V^y \\ V^z \end{pmatrix}$$

Then

$$\begin{pmatrix} \underline{V}^i \end{pmatrix} = \begin{pmatrix} R^i_{\ j} \end{pmatrix} \begin{pmatrix} V^i \end{pmatrix}$$

Vectors pg. 2

The rotation matrix

$$(R^{i}{}_{j}) = \begin{array}{cc} & j \longrightarrow \\ i \downarrow & \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rotations dont change norm

$$\underline{v}^T \underline{v} = v^T \underbrace{R^T R}_{\mathbb{1}} v = v^T v$$

So

$$R^{-1} = R^T$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R^T = R^{-1}$$

Then since $\vec{v}$ is unchanged under rotation we need that the basis change in opposite way

$$\underline{\vec{e}}_i = \vec{e}_i (R^{-1})^i{}_j \quad , \qquad \text{i.e.}$$

Basis vectors form a row:

$$(\underline{\vec{e}}_i \dots) = (\vec{e}_i \dots) \begin{pmatrix} R^{-1} \end{pmatrix}$$

So

$$\vec{V} = \underline{\vec{e}}_i \, \underline{v}^i$$

$$= (\vec{e}_i \dots) \underbrace{\begin{pmatrix} R^{-1} \end{pmatrix} \begin{pmatrix} R \end{pmatrix}}_{\mathbb{1}} \begin{pmatrix} v^i \\ \vdots \end{pmatrix}$$

$$= \vec{e}_i v^i$$

We used

$$(R^{-1})^i{}_j R^j{}_k = \delta^i{}_k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

- Contravariant / Covariant indices

For every set of upstairs indices (contravariant) (V^x, V^y, V^z) define the lowered (covariant) components (V_x, V_y, V_z) which transform as a row according to the inverse matrix

$$\boxed{\underline{V}_i = V_i (R^{-1})^i{}_j}$$

$$(\underline{V}_i \ldots) = (V_i \ldots) \begin{pmatrix} R^{-1} \end{pmatrix}$$

Now since $R^{-1} = R^T$ (since rotations preserve length i.e. $\underline{x}^T \underline{x} = x^T x$), we see that

$$V_i = V_i (R^T)^i{}_j = V_i (R)_j{}^i .$$

i.e

$$\underline{V}_i = (R)_j{}^i V_i$$

But this is the <u>same</u> transformation rule as for upstairs indices. So up and down are the <u>same</u> for rotations.

$$V_x = V^x, \quad \text{or} \quad V_i = \delta_{ij} V^j \qquad V^i = \delta^{ij} V_j$$

So indices are raised and lowered with δ^{ij}, + δ_{ij}

Covariant Contravariant pg. 2

Similary define contravariant basis vectors

$$\vec{e}^{\,i} = \delta^{ij}\,\vec{e}_j$$

Which transform as a column vector

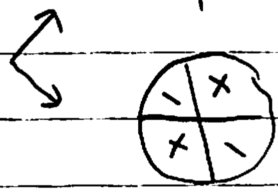
$$\underline{\vec{e}}^{\,i} = R^i{}_j\, e^j$$

So that the vector is rotationally invariant

$$\vec{V} = V_i\,\vec{e}^{\,i} = V^i\,\vec{e}_i$$

Tensors

- Example: Want to describe the anisotropy of the charge distribution, and its orientation. Sort of described by two vectors. We will see that the right concept is the quadrupole tensor



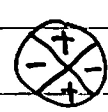
$$Q^{ij} = \int d^3x\, \rho(\vec{x}) \left(x^i x^j - \frac{1}{3} x^2 \delta^{ij} \right)$$

Rotations, rotate each arm of the tensor:

$$\underline{Q}^{ij} = R^i{}_l \, R^j{}_m \, Q^{lm}$$

rotated tensor

original tensor



$$\overleftrightarrow{Q} = Q^{ij} \, \vec{e}_i \vec{e}_j$$

Dot-Products and Cross-Products:

① $$\vec{a}\cdot\vec{b} = (a^i \vec{e}_i)(b^j \vec{e}_j) = a^i b^j \overbrace{\vec{e}_i \cdot \vec{e}_j}^{=\delta_{ij}}$$

$$= a^i b^j \delta_{ij} = a_i b^i$$

rotationally invariant. Prf. easy. Contracted indices are invariant

② To define cross product need the epsilon tensor

$$\varepsilon^{ijk} = \begin{cases} \pm 1 & \text{for } i,j,k \text{ even/odd perm of } 1,2,3 \\ 0 & \end{cases}$$

e.g $\varepsilon^{123} = \varepsilon^{312} = \varepsilon^{231} = \varepsilon^{123} = -\varepsilon^{213} = -\varepsilon^{321} = +1$

Then this is a tensor which is the same after rotation. Prf uses $\det R = 1$

$$\det R = R^1{}_i \, R^2{}_j \, R^3{}_k \, \varepsilon^{ijk}$$

$$= \begin{vmatrix} R^1{}_i & \cdots \\ R^2{}_j & \cdots \\ R^3{}_k & \cdots \end{vmatrix}$$

Cross Products pg. 2

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} = e_i \, \varepsilon^{ijk} a_j b_k$$

So

contravariant

$$(\vec{a} \times \vec{b})^i = \varepsilon^{ijk} a_j b_k = \text{i-th component of } \vec{a} \times \vec{b}$$

The

$$(a \times (b \times c))^i = \varepsilon^{ijk} a_j \varepsilon^{klm} b_l c_m$$

$$= \varepsilon^{ijk} \varepsilon^{klm} a_j b_l c_m$$

Think about it: for example, consider ε^{ij3}. then for ε^{ij3} is non-zero for $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$,

$$\varepsilon^{123} = -\varepsilon^{213} = 1$$

So thinking along these lines we conclude

$$\varepsilon^{ijk} \varepsilon^{klm} = \varepsilon^{ijk} \varepsilon^{lmk} = \delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}$$

So

$$a \times (b \times c)^i = (\delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}) a_j b_l c_m$$

$$= b^i (\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b}) c^i \qquad \text{"bac - abc" rule}$$

Derivative Operations:

$$\boxed{\partial_i \equiv \frac{\partial}{\partial x^i}}$$

grad $= (\nabla \vec{S})_i = \partial_i S$

curl $= (\nabla \times \vec{V})^i = \varepsilon^{ijk}\, \partial_j V_k$

div $= \nabla \cdot \vec{V} = \partial_i V^i = \partial_x V^x + \partial_y V^y + \partial_z V^z$

laplacian $\nabla \cdot \vec{\nabla} S = \partial_i \partial^i$

The $b(ac)-(ab)c$ rule plays an important role

$$\nabla \times (\nabla \times \vec{C}) = \vec{\nabla}(\nabla \cdot \vec{V}) - \nabla^2 \vec{C}$$

- Homework: use the $b(ac)-(ab)c$ rule to derive the wave eqn

Helmholtz Theorems

① If $\vec{\nabla}\cdot\vec{C} = 0$, then there exsists $\vec{D}$ such that:

$$\vec{C} = \vec{\nabla}\times\vec{D}$$

② If $\nabla\times\vec{C} = 0$, then there exsists a scalar field S such that

$$\vec{C} = \vec{\nabla} S$$

I wont prove it (but see homework) but I will show the converse, i.e.

① $\nabla\cdot(\vec{\nabla}\times\vec{D}) = 0$ and $\nabla\times(\vec{\nabla}S) = 0$

Prf

① $$\partial_i C^i = \partial_i \overbrace{\varepsilon^{ijk}\partial_j D_k}^{(\nabla\times C)^i} = \varepsilon^{ijk}\partial_i\partial_j D_k = 0$$

Because $\varepsilon^{ijk} = -\varepsilon^{jki}$ is antisymmetric while $\partial_i\partial_j = \partial_j\partial_i$ is symmetric

② Similarly

$$\underbrace{\varepsilon^{ijk}\partial_j C_k}_{(\nabla\times C)^i} = \varepsilon^{ijk}\partial_j\partial_k S = 0$$

These are statements of differential forms $ddD = 0$

Maxwell & The Helmholtz Thrms

$$\nabla \cdot E = \rho$$

$$\nabla \times B = j/c + \frac{1}{c}\partial_t E$$

$$\left.\begin{array}{ll} ① & \nabla \cdot B = 0 \\ ② & -\nabla \times E = \frac{1}{c}\partial_t B \end{array}\right\} \text{source free - constraint eqs}$$

Can "trivially" solve these two eqs. using Helmho

① $\nabla \cdot B = 0 \implies \boxed{\vec{B} = \vec{\nabla} \times \vec{A} \Leftarrow \vec{A} = \text{"vector potential"}}$

② Using $\vec{B} = \nabla \times \vec{A}$

$$-\nabla \times \vec{E} = \frac{1}{c}\partial_t \nabla \times \vec{A}$$

$$-\nabla \times \left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0$$

So we can write

scalar potential

$$\vec{E} + \frac{1}{c}\partial_t \vec{A} = -\nabla\phi$$

or

$$\boxed{\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\phi}$$