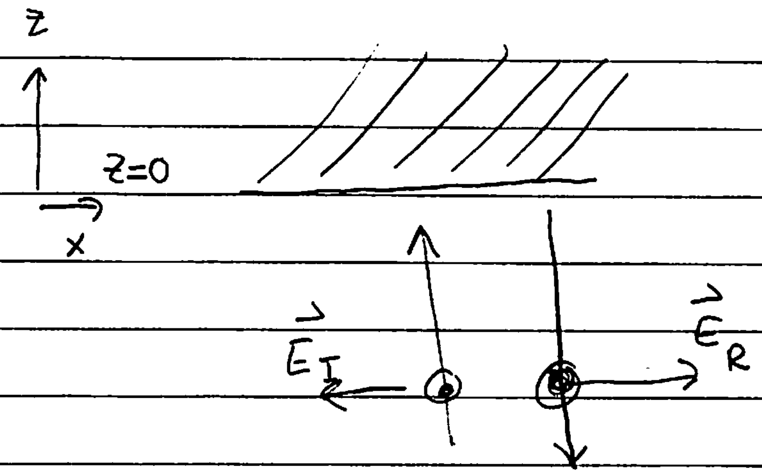
Reflection at Metal



- Metals are shiny. Why?
- Basic reason: induced currents do not allow e+m fields in metal
- At finite σ find corrections, wave penetrates a little

At $\sigma \to \infty$, find perfect reflection, $\vec{E}$ vanishes at interface, i.e.

$$\vec{E}_R = -\vec{E}_I \quad \text{does the job}$$

So,

$$\vec{E}_1 = \mathrm{Re}\left[\vec{E}_I e^{ikz} + \vec{E}_R e^{-ikz}\right]$$

$$= 2\vec{E}_I \sin(kz)$$

← vanishes at interface $z=0$

Find at finite σ, a small E_x and corresponding current

$$j_x = \sigma E_x$$

Maxwell Eqs in Metal

$$\nabla \cdot E = \rho_{mat}$$

$$\nabla \times B = \frac{\vec{j}_{mat}}{c} + \frac{1}{c} \partial_t \vec{E}$$

$$\nabla \cdot B = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c} \partial_t \vec{B}$$

no external charges or currents

Then

$$\vec{j}_{mat} = \sigma \vec{E} + \chi_e \partial_t \vec{E} + c \chi_m^B \nabla \times \vec{B}$$

$$\begin{aligned} j(\omega, \vec{x}) &= \sigma \vec{E}(\omega, x) + -i\omega \chi_e \vec{E} + c \chi_m^B \nabla \times \vec{B} \\ &= (i\sigma/\omega + \chi_e)(-i\omega \vec{E}) + c \chi_m^B \nabla \times B \end{aligned}$$

So from current consv.

$$\partial_t \rho = -\nabla \cdot \vec{j} \implies -i\omega \rho(\omega, x) = -\nabla \cdot \vec{j}$$

Find

$$\rho(\omega, x) = -\left(\frac{i\sigma}{\omega} + \chi_e\right) \nabla \cdot E + \frac{c \chi_m^B}{-i\omega} \cancelto{0}{\nabla \cdot (\nabla \times B)}$$

So

$$\nabla\cdot E = -\left(\frac{i\sigma}{\omega}+\chi_e\right)\nabla\cdot E$$

Compare to Dielectric

$$\Big(\underbrace{1+\chi_e}_{\varepsilon}+\frac{i\sigma}{\omega}\Big)\nabla\cdot E = 0$$

Thus, $\left(\varepsilon+\frac{i\sigma}{\omega}\right)\nabla\cdot E = 0$

$$\varepsilon\,\nabla\cdot E = 0$$

Similary

$$\nabla\times B = \frac{\vec{J}}{c} + \overbrace{-\frac{i\omega}{c}E}^{\frac{1}{c}\partial_t\vec{E}}$$

$$\nabla\times B = \frac{1}{c}\Big(\frac{i\sigma}{\omega}+\underbrace{\chi_e+1}_{=\varepsilon}\Big)(-i\omega E) + \chi_m^B\,\nabla\times B$$

So

$$\frac{1}{\mu}\nabla\times B = \frac{1}{c}\left(\frac{i\sigma}{\omega}+\varepsilon\right)(-i\omega)$$

with $\mu = 1/(1-\chi_m^B)$, so

Dielectric

$$\nabla\times\vec{B} = \frac{\mu}{c}\left(\frac{i\sigma}{\omega}+\varepsilon\right)(-i\omega\vec{E})$$

$$\nabla\times B = \frac{\mu\varepsilon}{c}(-i\omega E$$

Thus conclude

The maxwell eqs are the same with the replacement

$$\varepsilon \longrightarrow \boxed{\hat{\varepsilon}(\omega) = \varepsilon + i\sigma/\omega}$$

i.e

$$\hat{\varepsilon} \; \nabla \cdot E = 0$$

$$\nabla \times B = \frac{\mu \hat{\varepsilon}}{c} (-i\omega \vec{E})$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = \frac{i\omega}{c} \vec{B}$$

Note: that $\sigma \sim 10^{18}$ 1/s $\omega \sim$ GHz so

$\frac{\sigma}{\omega} \gg 1$, while $\varepsilon \sim 1$ thus

to a good approximation $\hat{\varepsilon}(\omega) \simeq \frac{i\sigma}{\omega}$

Wave - Solutions in Metal

- Try a solution $\vec{H}(\vec{x}) = \vec{H}_T e^{i\vec{k}\vec{x}}$ in Helmholtz eqn

$$\left(\nabla^2 + \frac{\omega^2 \mu \hat{\varepsilon}}{c^2} \right) \vec{H}(x) = 0$$

So

$$-k^2 + \frac{\omega^2 \mu}{c} \left(\frac{i\sigma}{\omega} \right) = 0$$

Find

$$k = \pm \sqrt{\frac{\omega \mu \sigma}{c^2}} \sqrt{i} = \sqrt{\frac{\omega \mu \sigma}{2c^2}} (1+i)$$

i.e

$$k = \pm \left(\frac{1+i}{\delta} \right) \qquad \boxed{\delta = \left(\frac{2c^2}{\omega \mu \sigma} \right)^{1/2}}$$

So find then (selecting + sign for decreasing exponent)

$$\vec{H}_T(x) = \vec{\mathcal{H}}_T e^{-z/\delta} e^{iz/\delta} = \mathrm{Re}\, \mathcal{H}_T e^{ik \cdot x}$$

Now lets look at $\boxed{\vec{E} \equiv \vec{E}_T e^{ik \cdot x}}$

$$\nabla \times H = \frac{\sigma E}{c}$$

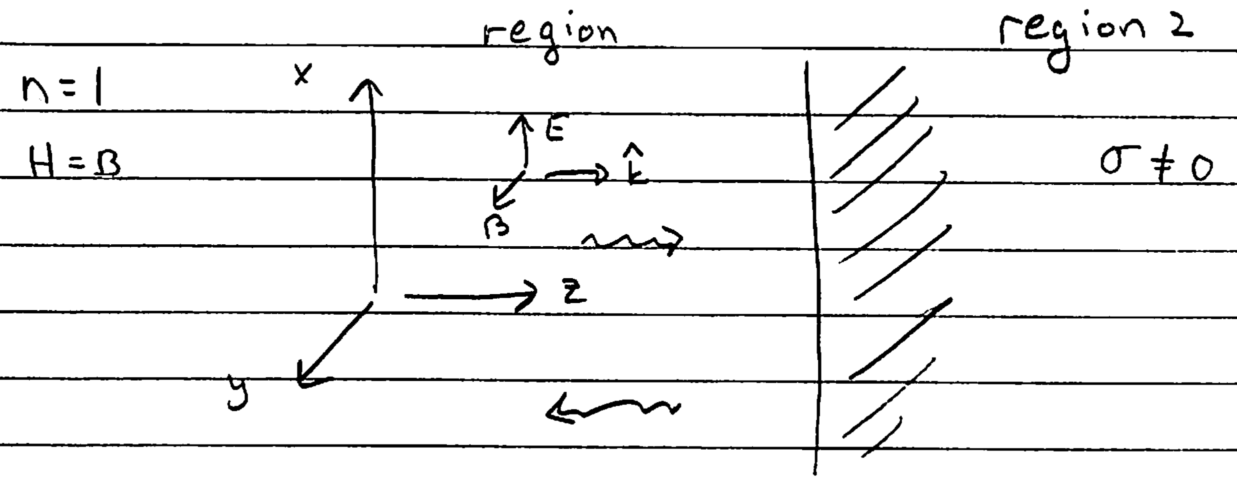
or $\quad i\vec{k} \times \vec{H} = \sigma/_c \vec{E} \Rightarrow i \frac{(1+i)}{\delta} \hat{n} \times \vec{\mathcal{H}}_T = \frac{\sigma}{c} \vec{E}_T$

Thus with $\delta = \sqrt{\frac{2c^2}{\mu\omega\sigma}}$

$$\boxed{\vec{E}_T = -\sqrt{\frac{\mu\omega}{\sigma}}\,\frac{(1-i)}{\sqrt{2}}\,\vec{n}\times\vec{H}_T}$$

Note then that since $\sigma \gg \omega$, we have $E \ll H$ in metal

Matching at the vacuum-metal interface



Region 1 - vacuum

$$\vec{E}_1 = (E_I e^{ikz} + E_R e^{-ikz})\ \hat{x}$$

$$\vec{H}_1 = (H_I e^{ikz} + H_R e^{-ikz})\ \hat{y}$$

given $\vec{E} = (-\hat{k}) \times \vec{H}$, $\quad E_R = -H_R$

$$\vec{E} = (H_I e^{ikz} - H_R e^{-ikz})\ \hat{x}$$

Region 2

$$\vec{H}_2 = \vec{H}_c e^{ik_T z} \equiv H_c e^{-z/\delta} e^{iz/\delta}\ \hat{y}$$

$$E_2 = \sqrt{\frac{\mu\omega}{\sigma}}\ \frac{(1-i)}{\sqrt{2}}\ H_c e^{ik_T z}\ \hat{x}$$

Matching

$H_{\parallel}$ is conituous : $\quad n \times (\vec{H}_{out} - \vec{H}_{in}) = 0$

(1) $\quad H_c - (H_I + H_R) = 0$

$E_{\parallel}$ is continuous: $\quad n \times (\vec{E}_{out} - \vec{E}_{in}) =$

(2) $$\underbrace{\sqrt{\frac{\mu\omega}{\sigma}}\,\frac{(1-i)}{\sqrt{2}}\,H_c}_{} - \underbrace{(H_I - H_R)}_{E \text{ in vacuum}} = 0$$

$E_c \ll H_c$

Small $\omega \ll \sigma$

Solve order by order (although you can do it exactly

<u>0th</u> $\quad H_I = H_R \quad$ and $\quad H_c = 2H_I$

so $\quad E_I = H_I = H_R = -E_R$

find perfect reflecton

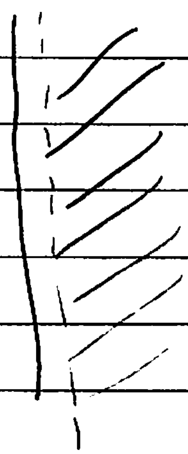
<u>1st</u> Can use $H_c \simeq 2H_I$ in Eq.(2) and solve for H_R since it multiplies $\sqrt{\frac{\omega}{\sigma}}$

$$H_R = H_I - 2\sqrt{\frac{\mu\omega}{\sigma}}\,\frac{(1-i)}{\sqrt{2}}\,H_I$$

Now Eq (1) gives

$$H_c = H_I + H_R = 2H_I - \underbrace{2\sqrt{\frac{\mu\omega}{\sigma}}\,\frac{(1-i)}{\sqrt{2}}\,H_I}_{\text{small}}$$

Energy Flow intor metal



Evaluate the Poynting flux just inside metal

$$\vec{S}_c\cdot\vec{n}\Big|_{z=0} = \frac{c}{2}\,\mathrm{Re}\;\overset{\text{small}}{E}\times H^*$$

$$H = H_c\, e^{ik_T z}\Big|_{z=0} = H_c = 2H_I + \text{small}$$

$$E = \mathrm{Re}\sqrt{\frac{\mu\omega}{\sigma}}\,\frac{(1-i)}{\sqrt{2}}\,H_c\, e^{ik_T z}\Big|_{z=0} = \sqrt{\frac{\mu\omega}{\sigma}}\,\frac{1}{\sqrt{2}}\cdot 2H_I$$

So

$$S_{loss} = \vec{S}_c \cdot \vec{n} = c\sqrt{\frac{\mu\omega}{\sigma}}\ \frac{2\cdot 2}{2}\ \frac{1}{\sqrt{2}}\ H_I^2\ \hat{z}\cdot\hat{z}$$

$$= c\sqrt{\frac{2\mu\omega}{\sigma}}\ |H_I|^2$$

The input energy is:

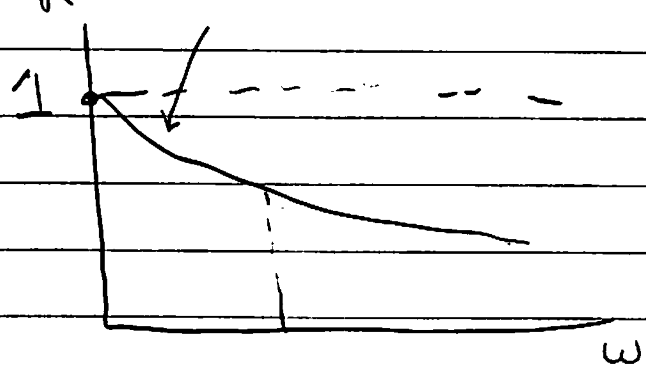
$$S_{in} = \frac{c}{2}\,|H_I|^2$$

So

$$R = \frac{S_{in} - S_{loss}}{S_{in}} = 1 - \frac{S_{loss}}{S_{in}}$$

$$R \simeq 1 - 2\sqrt{\frac{2\mu\omega}{\sigma}}$$

R describes the reflection coefficient here



Energy dissipated in Conductor - Direct Calculation pg. 1

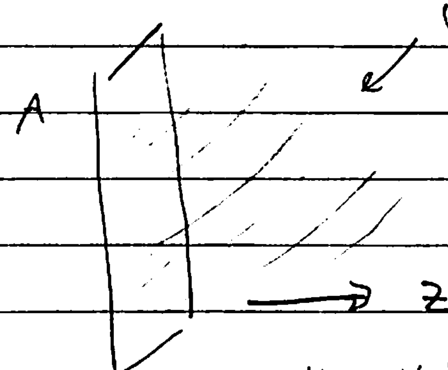
$$\left\langle \frac{dW}{dt} \right\rangle_{\text{time ave}} = \int_V d^3x \, \frac{Re(\vec{j}^{\,*} \cdot \vec{E})}{2}$$

for a time ave take half Re part

$\vec{j} = \sigma \vec{E}$

$$= \frac{1}{2} \sigma \int_V d^3x \, |E|^2$$

$$= \frac{1}{2} \sigma \underbrace{A}_{\text{area}} \int_0^\infty dz \, |E^2|$$



Volume of metal we are integrating

H_c Value of H at interface $\simeq 2H_I$

Using

$$|E_c|^2 = \frac{\mu\omega}{\sigma} |H_c|^2 e^{-2z/\delta}$$

do integral use $\delta = \sqrt{\frac{2c^2}{\mu\sigma\omega}}$

So

$$\boxed{\frac{1}{A} \left\langle \frac{dW}{dt} \right\rangle} = \frac{1}{2} \sigma \int_0^\infty dz \, \frac{\mu\omega}{\sigma} 4 H_I^2 e^{-2z/\delta} = \boxed{c \sqrt{\frac{2\mu\omega}{\sigma}} \, |H_I|^2}$$

This agrees ⓦ S_{loss} the flux into metal.