Last Time

- Started to describe the retarded Grn function of SHO

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2\right] G_R(t,t_0) = \delta(t-t_0)$$

- Demand causality

$$G_R(t,t_0) = 0 \quad \text{for } t < t_0$$

- Showed that the inhomogeneos solution is

$$x(t) = \int_{-\infty}^{\infty} G_R(t,t_0)\, F(t_0)\, dt_0 \Rightarrow x(\omega) = G_R(\omega) F$$

- All physical quantities are ultimately expressible as Grn-fcns. For example, we used a damped harmonic oscillator (Lorentz model) to describe the dielectric constant. $\vec{F} = eE(t)$ (force)
  The current $\vec{j}(t) = Nev(t)$, so:

$$j(\omega) = Ne(-i\omega\, x(\omega))$$

$$x(\omega) = G_R(\omega) F(\omega)$$

$$j(\omega) = Ne^2(-i\omega\, G_R(\omega))E(\omega)$$

constitutive relation

Comparison, $j(\omega) = -i\omega \chi_e(\omega) E(\omega)$ shows that

$$\chi_e(\omega) = Ne^2 G_R(\omega)$$

Thus we see how in a particular model the response function of the dynamics determines the susceptibility

Find $G_R(t)$ in time: Direct method

$$\left[m \frac{d^2}{dt^2} + m\eta \frac{d}{dt} + m\omega_0^2 \right] G_R(t, t_0) = \delta(t - t_0)$$

Demand continuity and integrating from $t_0 - \varepsilon$, to $t_0 + \varepsilon$. We know $G_R(t, t_0) = 0$ for $t < t_0$

$\star$ $\quad G(t_0 + \varepsilon, t_0) = 0$ $\qquad$ Causality

$(\star\star)$ $\quad m\partial_t G(t + \varepsilon, t_0) = 1$ since $m\partial_t G(t+\varepsilon, t_0) - \cancel{m\partial_t G(t_0 - \varepsilon, t_0)}^{0}$

Also note that if one differentiates $(\star\star)$ with respect to t_0 $\qquad = \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \delta(t - t_0) = 1$

$$\partial_{t_0} \partial_t G(t + \varepsilon, t_0) = 0$$

$m\partial_t G(t+\varepsilon, t_0)$

$t_0 \qquad t = t_0 + \text{bit}$

Then we can solve the diff-eq given the initial conditions. The two homogeneous solutions are

$$x_\pm(t) = e^{-\frac{\eta}{2}t} e^{\pm i\omega_0 t} \qquad \text{for small } \eta$$

Then the linear combo which satisfies the boundary conditions $\star$ and $\star\star$ is

$$G_R(t, t_0) = \begin{cases} \dfrac{\sin \omega_0 (t - t_0)\, e^{-\eta/2 (t - t_0)}}{m\omega_0} & t - t_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

Usually written

$$G_R(\tau) = \Theta(\tau)\,\frac{\sin\omega_0\tau}{m\omega_0}\,e^{-\eta/2\,\tau} \qquad \tau \equiv t - t_0$$

Fourier Method for Green fcn, $\int \frac{d\omega}{2\pi} e^{-i\omega\tau}$

$$\left[m\frac{d^2}{d\tau^2} + m\eta\frac{d}{d\tau} + m\omega_0^2\right] G_R(\tau) = \delta(\tau)$$

Fourier Transform both sides

$$\left[-m\omega^2 + m\eta(-i\omega) + m\omega_0^2\right] G_R(\omega) = 1$$

$$G_R(\omega) = \frac{1/m}{\left[-\omega^2 + \omega_0^2 - i\omega\eta\right]}$$

Thus

$$G_R(\tau) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}\, 1/m}{\left[-\omega^2+\omega_0^2 - i\omega\eta\right]}$$

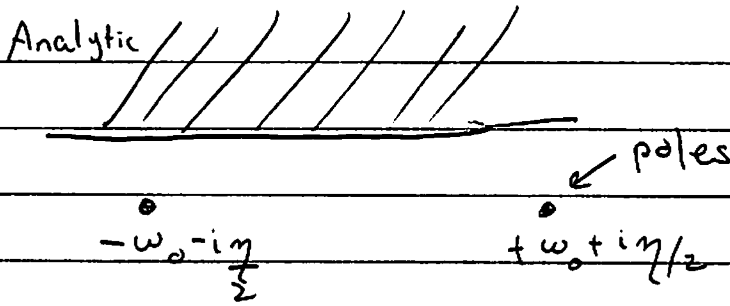
You can do these integrals with contour integration the poles are at

$$\omega^2 + i\omega\eta = \omega_0^2$$

Solving this equation for small η:

$$\omega \simeq \pm\omega_0 - i\frac{\eta}{2}$$

We see that the integrand has the following analytic structure



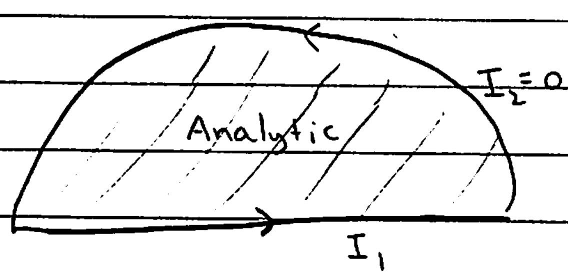
So now we should do the integral:

Case 1: $\tau < 0$ $\quad G_R(\tau) = 0 \leftarrow$ causality

The math works like this, since $\tau < 0$:

$$e^{-i\omega\tau} \xrightarrow[\omega \to \text{complex}]{} e^{-i\,\mathrm{Re}\,\omega\,\tau}\,\underbrace{e^{+[\mathrm{Im}\,\omega]\tau}}_{\substack{\text{decreasing exponentially} \\ \text{for } \mathrm{Im}\,\omega > 0}} \quad \leftarrow \tau < 0$$

Thus for $\tau < 0$ we can close the contour in the UHP without picking up poles and find zero



$$I_1 + \cancelto{0}{I_2} = 0$$

$$I_1 = 0$$

Case 2: $\tau > 0$

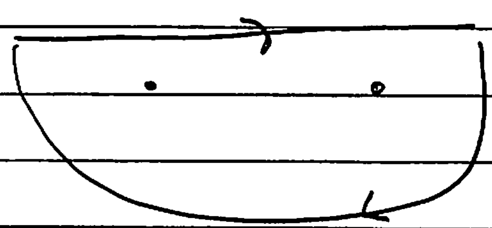
For $\tau > 0$ we must close the contour in the LHP picking up poles at $\omega = \pm\omega_0 - i\frac{\eta}{2}$

For $\tau > 0$: wrong way arand poles

$$G_R(\tau) = -2\pi i \left[\mathrm{Res}_{\omega=\omega_0 - i\frac{\eta}{2}} + \mathrm{Res}_{\omega=-\omega_0 - i\eta/2} \right]$$

$$= \frac{1}{m}\frac{i}{2\omega_0} e^{-\eta/2 t} e^{-i\omega_0\tau} + \frac{1}{m}\frac{-i}{2\omega_0} e^{-\eta/2\tau} e^{+i\omega_0\tau}$$

homogeneous solutions from two pages back



$$= \frac{1}{m} e^{-\eta/2\tau} \frac{\sin\omega_0\tau}{\omega_0}$$

So

$$G_R(\tau) = \Theta(\tau)\frac{\sin\omega_0\tau}{m\omega_0} e^{-\eta/2\tau} \xrightarrow[\eta\to 0]{} \Theta(\tau)\frac{\sin\omega_0\tau}{m\omega_0}$$

We will see that this Green function is closely related to the green function of the wave eqn

(Aside. $i\varepsilon$ perscription :)

Take the $\eta \to 0$ limit of the damped harmonic oscillator

$$G_R(\tau) = \frac{\sin \omega_0 \tau}{m\omega_0} \, \Theta(\tau)$$

$$G_R(\omega) = \frac{1/m}{[-\omega^2 + \omega_0^2]}$$

But this is ambiguous since the poles are on the real axis. What does this mean $\int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{(-\omega^2+\omega_0^2)}$?

We know that causality demands that the poles lie in the lower half plane. We can enforce this by adding an infinitessimal imaginary part

$$\omega \to \omega + i\varepsilon$$ positive

So

$$G_R(\omega) = \frac{1/m}{(-(\omega+i\varepsilon)^2 + \omega_0^2)}$$

$$= \frac{1/m}{(-\omega^2 + \omega_0^2 - 2i\varepsilon\omega)}$$

Amounts to adding an infinitessimal damping coeffici[ent]

$\eta = 2\varepsilon$

Causality and Analyticity

- We saw in a particular model that the analyticity in the UHP was essential to getting the $G(\tau) = 0$ for $\tau < 0$. In fact any causal function will be analytic in the UHP

$$G(\omega) = \int_{-\infty}^{\infty} d\tau \, e^{+i\omega\tau} G(\tau) = \int_{0}^{\infty} e^{i\omega\tau} G(\tau)$$

causality

Looking at the second integral, the exponent

$$e^{i\omega\tau} \longrightarrow e^{i\operatorname{Re}\omega} e^{-\operatorname{Im}\omega\,\tau} \leftarrow \tau > 0$$

Decreases as ω is extended into the complex plane. Thus the fourier integral provides an analytic continuation into the UHP, defining $G(\omega)$ uniquely.

- For such causal analytic functions the Re parts and imaginary parts are related; (Kramers Kronig Relation)

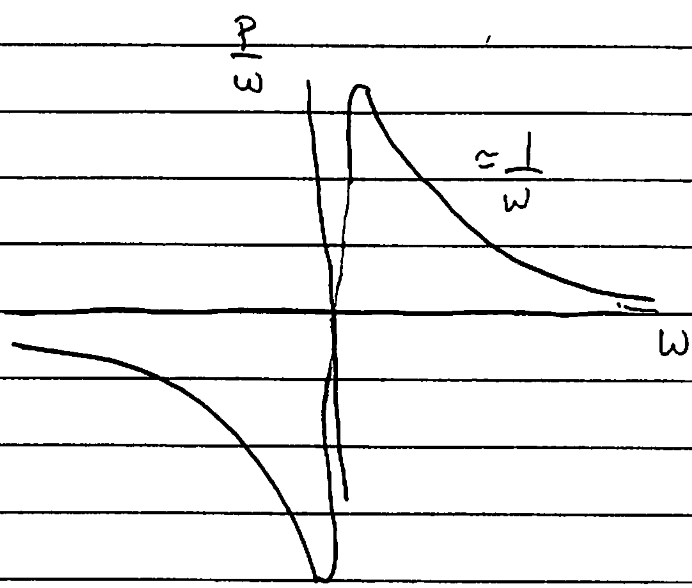
$$\begin{cases} \operatorname{Re} G_R(\omega) = -\displaystyle\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{P}{\omega-\omega'} \operatorname{Im} G_R(\omega') \\ \operatorname{Im} G_R(\omega) = \displaystyle\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{P}{\omega-\omega'} \operatorname{Re} G_R(\omega') \end{cases}$$

Stated without proof

Here $\frac{P}{\omega - \omega_0}$ is the principal value function. Much

like the δ-fcn it should be thought of as a limit of a sequence of functions which are integrated over

$$\frac{P}{\omega-\omega_0} = \lim_{\varepsilon \to 0} \frac{(\omega-\omega_0)}{(\omega-\omega_0)^2+\varepsilon^2}$$

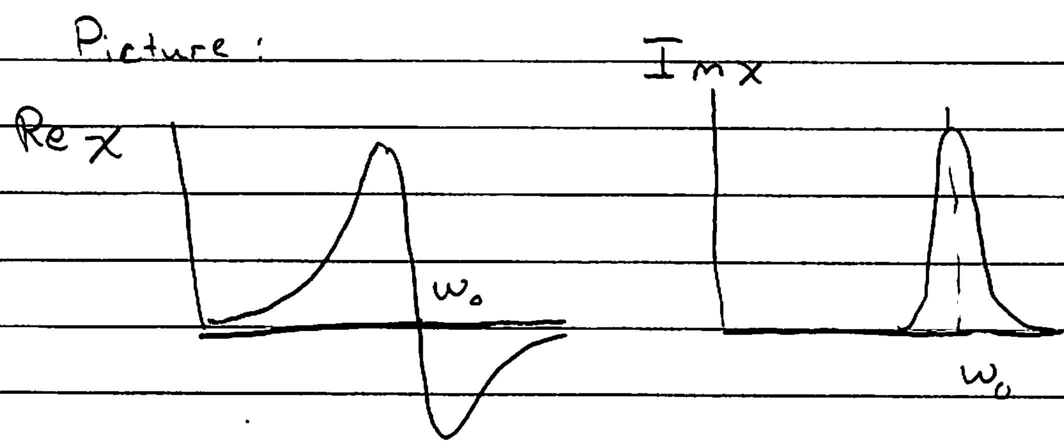


- Let us see how causality determines the qualitative features of the dielectric constant

~~Lorent~~ model

$$\chi_e(\omega) = \underbrace{\frac{Ne^2/m\,(-\omega^2+\omega_0^2)}{[(\omega^2-\omega_0^2)^2+(\omega\eta)^2]}}_{\text{real part}} + i\,\underbrace{\frac{Ne^2/m\;\omega\eta}{[(\omega^2-\omega_0^2)^2+(\omega\eta)^2]}}_{\text{Im part}}$$

Picture:



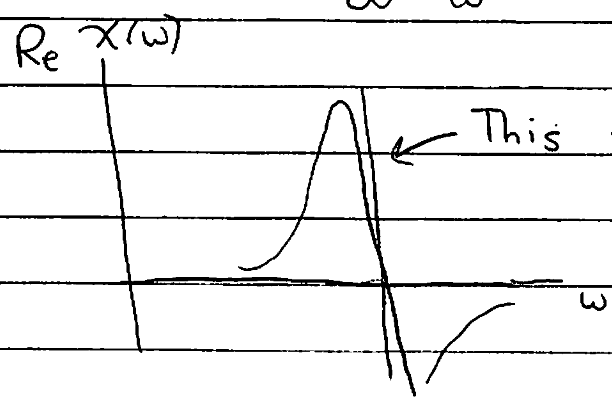
We saw that this functional form gave a good qualitative description of real dielectric functions.

We can now see why. If $\operatorname{Im}\chi(\omega) = C\pi\,\delta(\omega-\omega_*)$ + regular has an absorption band at $\omega \approx \omega_*$, then:

$$\operatorname{Re}\chi(\omega) = -P\int \frac{d\omega'}{\pi}\,\frac{\operatorname{Im}\chi(\omega')}{\omega-\omega'}$$

$$\operatorname{Re}\chi(\omega) = -P\int \frac{d\omega'}{\pi}\,\frac{C\pi\,\delta(\omega'-\omega_*)}{\omega-\omega'}$$

$$\operatorname{Re}\chi(\omega) \simeq -P\,\frac{1}{\omega-\omega^*}$$

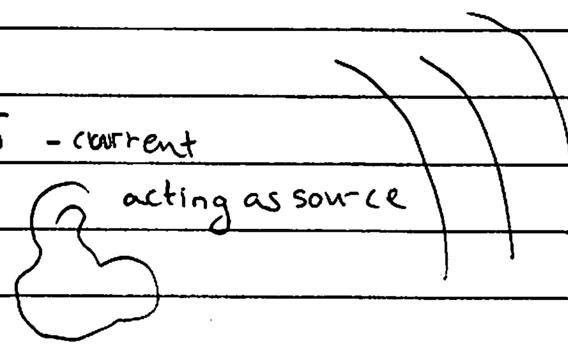


This feature is determined by causality in the presence of an absorption band

Green Fcn of the Wave-Eqn

$$-\Box u(t,x) = J(t,x)$$ ← source. In E+M these will be currents

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$



The induced waves are

$$u(t,x) = \int dt_0\, dx_0\, G_R(t-t_0, \vec{x}-\vec{x}_0)\, J(t_0,x_0)$$

Then $G_R(t\vec{x}\,|\,t_0 x_0)$ is the field at $t\vec{x}$ due to a point charge at $t_0 x_0$

$$-\Box G_R(t,\vec{x}\,|\,t_0 x_0) = \delta(t-t_0)\,\delta^3(x-x_0)$$

Then:

$$-\Box u(t,x) = \int \overbrace{-\Box G_R(tx\,|\,t_0 x_0)}^{\delta(t-t_0)\delta^3(x-x_0)}\, J(t_0,x_0)$$

$$-\Box u(t,x) = J(t,x) \quad ✓$$

Solving for the Grn-fcn

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(t-t_0, \vec{x}-\vec{x}_0) = \delta(t-t_0)\,\delta^3(\vec{x}-\vec{x}_0)$$

First choose $t_0, x_0 = 0$:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] G(t,\vec{x}) = \delta(t)\,\delta^3(\vec{x})$$

Now Fourier transform in Space: $G(t,\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}}\, G(t,x)$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + k^2\right] G(t,k) = \delta(t)$$

or

$$\frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2} + (ck)^2\right] G(t,k) = \delta(t)$$

Compare to SHO :

$$m\left[\frac{\partial^2}{\partial t^2} + \omega_0^2\right] G(t) = \delta(t)$$

Thus we can take the SHO result "lock-stock + barrel"
with $\omega_0 = ck$ and $m \to 1/c^2$

$$G(\tau, k) = c^2\,\Theta(\tau)\,\frac{\sin(ck\tau_0)}{ck}$$

Now we "only" need to take the inverse FT:

$$G(\tau, \vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} c^2 \Theta(\tau) \frac{\sin ck\tau}{ck}$$

The integral is not convergent. But this is not a surprise. Add a convergence factor

$$G_\varepsilon(t, \vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{-\varepsilon|k|} e^{i\vec{k}\cdot\vec{r}} c^2 \Theta(\tau) \frac{\sin ck\tau}{ck}$$

Think of $G(t,\vec{r})$ as a limit of sequence, $G_\varepsilon(t,\vec{r})$, of functions which satisfy

$$-\Box G_\varepsilon(t,r) = \delta(t)\, \delta^3_\varepsilon(\vec{r})$$

A dirac sequence

$$\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} e^{-\varepsilon|k|}$$

To do the integral write $R = |\vec{r}|$

$$G_\varepsilon(t,\vec{r}) = \int \frac{k^2 dk\, d(\cos\theta)\, d\phi}{(2\pi)^3} e^{-\varepsilon k} e^{ikR\cos\theta} c^2 \Theta(\tau) \frac{\sin ck\tau}{ck}$$

Doing angular integral $\int_{-1}^{1} d(\cos\theta)\, e^{ikR\cos\theta} = \frac{2\sin kR}{kR}$

we find

$$G_\varepsilon(t,\vec{r}) = \frac{1}{2\pi^2} \frac{c\Theta(\tau)}{R} \int_0^\infty e^{-\varepsilon k} \sin kR \, \sin ck\tau$$

The remaining integrals can be done writing

$$\sin kR \sin ck\tau = \frac{1}{2}\left[\cos\left(k(R-c\tau)\right) + \cos\left(k(R+c\tau)\right)\right]$$

and with $\cos k(R-c\tau) = \frac{1}{2}\left[e^{ik(R-c\tau)} + e^{-ik(R-c\tau)}\right]$

So that

$$\int_0^\infty dk\; e^{-\varepsilon k} \cos\left(k(R-c\tau)\right) = \frac{\varepsilon}{\left[(R-c\tau)^2+\varepsilon^2\right]}$$

We find

$$G_\varepsilon = \frac{1}{4\pi R}\, c\,\Theta(\tau)\left[\frac{1}{\pi}\frac{\varepsilon}{(R-c\tau)^2+\varepsilon^2} + \frac{1}{\pi}\frac{\varepsilon}{(R+c\tau)^2+\varepsilon^2}\right]$$

Using

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} = \delta(x)$$

Find

can't be satisfied $R>0,\ \tau>0$

$$G(\tau,R) = \frac{c}{4\pi R}\,\Theta(\tau)\left[\underbrace{\delta(R-c\tau)}_{\substack{\text{pull out } c \\ \frac{1}{c}\delta\left(\frac{R}{c}-\tau\right)}} + \delta(R+c\tau)\right]$$

$$\boxed{G(\tau,R) = \frac{1}{4\pi R}\,\Theta(\tau)\,\delta\left(\frac{R}{c}-\tau\right)}$$

More generally

$$G(t-t_0, \vec{r}-\vec{r}_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\,\theta(t-t_0)\,\delta\left(\frac{|\vec{r}-\vec{r}_0|}{c} - (t-t_0)\right)$$

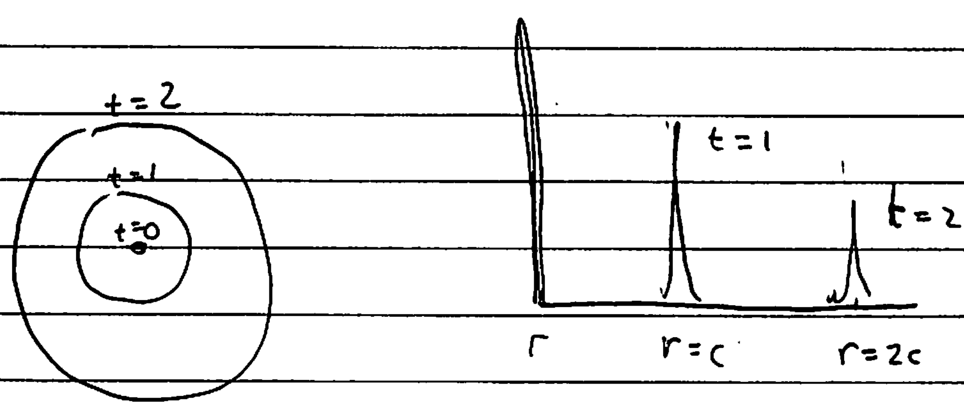
So

$$u(t,x) = \int dt_0\, d^3\vec{r}_0\; G(t-t_0, \vec{r}-\vec{r}_0)\, J(t_0, \vec{r}_0)$$

$$\boxed{u(t,x) = \int d^3\vec{r}_0\, \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\, J\left(t - \frac{|\vec{r}-\vec{r}_0|}{c}, \vec{r}_0\right)}$$

retarded time

Picture:



Also record

$$\boxed{G_R(\omega,k) = \frac{c^2}{(-(\omega+i\epsilon)^2 + (ck)^2)}}$$

Summary

$$G_R(\tau, R) = \frac{\Theta(\tau)}{4\pi R}\,\delta\left(\frac{R}{c} - \tau\right)$$

$$G_R(\tau, \vec{k}) = c^2\,\Theta(\tau)\,\frac{\sin(ck\tau)}{ck}$$

$$G_R(\omega, k) = \frac{c^2}{\left[-(\omega + i\varepsilon)^2 + (ck)^2\right]}$$