1. \[ \nabla \cdot \mathbf{E} = \rho \] 
\[ \nabla \times \mathbf{B} = \mathbf{J}/c + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \] 
\[ \nabla \cdot \mathbf{B} = 0 \] 
\[ -\nabla \times \mathbf{E} = \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \] 
\[ \nabla \cdot \mathbf{A} = \mathbf{J}/c \] 
\[ \mathbf{B} = \nabla \times \mathbf{A} \]
\[ \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi \]

2. Waves (in Lorentz gauge \( \nabla \cdot \mathbf{A} = 0 \))

\[ -\nabla^2 \varphi = \rho \]

3. Solve using Green's function:

\[ \mathbf{A}(t, \mathbf{r}) = \int d^3r_0 \frac{1}{4\pi |\mathbf{r} - \mathbf{r}_0|} \mathbf{J}(T, \mathbf{r}_0) \]

\[ T = t - \frac{1}{c} |\mathbf{r} - \mathbf{r}_0| \] \( \leftarrow \) retarded time

\[ (t, \mathbf{r}) = \text{observation point} \]

\[ \mathbf{n} = \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|} \]

\[ \mathbf{n} = \text{source point} \]
At large distances can approximate, even for highly relativistic sources, \( \Gamma \gg \lambda_{\text{typ}} \sim cT_{\text{typ}} \)

\[
\hat{A}_{\text{rad}} = \frac{1}{4\pi} \int \frac{1}{c^2} \frac{1}{c^2} \left( \begin{array}{c} t - r - \hat{n} \cdot \hat{r}_0 \end{array} \right) d^3r_0
\]

i.e.

\[
T = t - \frac{\hat{n} \cdot \hat{r}_0}{c}
\]

\[
E_{\text{rad}} = \hat{n} \times \hat{\mathbf{n}} \times \frac{1}{c} \frac{d}{dt} \hat{A}_{\text{rad}}
\]

\[
B_{\text{rad}} = -\hat{n} \times \frac{1}{c} \frac{d}{dt} \hat{A}_{\text{rad}}
\]

4. Then we concentrated first on non-rel sources where \( L \ll \frac{c}{T_{\text{typ}}} \) or \( L \ll \lambda_{\text{typ}} \). So far non-rel source, this is the picture.

(near zone) \( \Gamma \ll \lambda_{\text{typ}} \)

(far zone) \( \Gamma \gg \lambda_{\text{typ}} \)

What we studied in quasi-statics
For non-relativistic sources $n \cdot \frac{\hat{r}}{c}$ is small compared to $t$.

Thus in a non-relativistic approximation we write:

$$\frac{\partial}{\partial \tau} \left( \frac{\hat{r} - \frac{\hat{r}}{c} + n \cdot \hat{r_0}}{c} \right) = \frac{\partial}{\partial \tau} \left( \frac{\hat{r} - \frac{\hat{r}}{c}}{c} \right) + \frac{n \cdot \hat{r_0}}{c} \frac{\partial}{\partial t} \left( \frac{\hat{r} - \frac{\hat{r}}{c}}{c} \right) + \ldots$$

So we define $t_e = t - \frac{r}{c} = \text{emission time}$ to save writing:

5) Two Examples So far:

a) Radiation from a charged particle. In this case one has simply:

$$\int \frac{J(t_e) \, d^3r}{c} = \frac{e \cdot V(t_e)}{c}$$

and

$$A_{\text{rad}} = \frac{e}{4 \pi r} \frac{V(t_e)}{c} \quad \text{transverse piece of} \quad A$$

$$E_{\text{rad}} = n \times n \times z \left( \frac{1}{c} \frac{\partial A_{\text{rad}}}{\partial t} \right) = \frac{e}{c} \frac{-\hat{a}_e(t_e)}{c^2}$$
a) continued... leading to the power radiated
\[ P = \int r^2 d\Omega \cdot c(E \times B) \]

\[
P = \frac{e^2}{4\pi} \frac{2}{3} \frac{a^3(t_e)}{c^3}
\]

Larmor formula for the power radiated

b) Multipole expansion of localized source

\[ \dot{J}(t - \frac{\mathbf{r}}{c}) = J(t_e) + \mathbf{\nabla} \times \mathbf{\nabla} \cdot \mathbf{r} \cdot \mathbf{J} + \ldots \]

\[ \text{magnetic dipole} \quad \text{and quadrupole approx, today} \]

First we had the electric dipole:
\[
\mathbf{A}_{\text{rad}} = \frac{1}{4\pi r} \int \frac{\dot{J}(t_e)}{c} \cdot \mathbf{r} \left[ \frac{\mathbf{r}}{r^2} \right] 
\]

\[ \mathbf{J} = \nabla \times \mathbf{P} \leftarrow \text{capitol } \mathbf{P} \text{ is the dipole moment/Volume,}
\]

integrating over volume gives the dipole moment \( \mathbf{P} \)
Then note that this forms a spherical wave. Take \( \hat{p}(t) = p_0 e^{-i \omega t} \), then since we evaluate, \( t = t - \frac{r}{c} \):

\[
\hat{A}_{rad} = -i \omega p_0 e^{-i \omega (t - \frac{r}{c})}
\]

outgoing spherical wave.

Then we have:

\[
E_{rad} = \frac{n \times n \times 1}{c} \frac{dA_{rad}}{dt} = \frac{1}{4 \pi r} \left[ -\frac{\hat{p}(t)}{c^2} \right]
\]

After computing the power, we find for a harmonic source, \( P = p_0 e^{-i \omega t} \):

\[
\langle \frac{dR}{dt} \rangle = c \left| E_{rad} \right|^2
\]

Then we found a radiation pattern shown to the left. The characteristic features are:

1. \( P \propto \omega^4 \)
2. \( \sin^2 \theta \)
3. Polarization
Magnetic Dipole (M1) + Electric Quadrupole (E2)

Now we can't continue with the expansion:

\[ J\left(t - \frac{\mathbf{r}}{c}, \mathbf{r}_0\right) \approx J\left(t_0, \mathbf{r}_0\right) + \frac{\mathbf{r}}{c} \frac{\partial J\left(t_0, \mathbf{r}_0\right)}{\partial t} + \ldots \]

So the next term gives:

\[ \mathbf{A}^{\text{rad}} = \frac{1}{4\pi\mathbf{r}^3} \int \frac{\mathbf{n} \cdot \mathbf{r}}{c} \frac{\partial J\left(t - \frac{\mathbf{r}}{c}, \mathbf{r}_0\right)}{\partial t} \, d\mathbf{r}_0 \]

\[ \mathbf{A}^{\text{dip}} = \frac{n_0}{4\pi\mathbf{r}^3} \int \frac{\mathbf{r} \cdot \mathbf{r}}{c} \frac{\partial J\left(t_0, \mathbf{r}_0\right)}{\partial t} \, d\mathbf{r}_0 \]

As always, tensors \( \mathbf{r} \cdot \partial J/\partial t \) should be broken up into its irreducible components and analyzed separately. We will see that each irreducible component gives:

\[ \mathbf{r} \cdot \partial J/\partial t = \frac{1}{2} \left( \mathbf{r} \cdot \partial J + \mathbf{r} \times \frac{\partial \mathbf{J}}{\partial t} \right) \mathbf{j} - 2 \mathbf{J} \times \frac{\partial \mathbf{J}}{\partial t} + \frac{1}{3} \epsilon_{ijk} (\mathbf{r} \times \partial \mathbf{J}) \]

\[ \text{Quadrupole \ rad} + \text{magnetic dipole} \]

We will first analyze the magnetic dipole case. The general case is a sum of these + e-dipole term

The monopole, gives nothing; a monopole doesn't radiate
Magnetic Dipole pg 2

For magnetic dipole (sign reversed because I reversed i,j relative to last page)

\[
\vec{A}_{rad} = \frac{1}{4\pi r^3 c} \int \frac{1}{2} \mathbf{\hat{n}} \cdot (\mathbf{r}_0 \times \frac{\mathbf{\dot{J}}(t_c, \mathbf{r}_0)}{c}) d\mathbf{r}_0
\]

\[
\vec{A}_{rad} = -\frac{1}{4\pi r c} \mathbf{\hat{n}} \times \frac{1}{2} \int \frac{\mathbf{r}_0 \times \mathbf{\dot{J}}(t_c, \mathbf{r}_0)}{c} d\mathbf{r}_0
\]

we defined \( \bar{m} = \frac{1}{2} \int \mathbf{r}_0 \times \frac{\dot{\mathbf{J}}}{c} d\mathbf{r}_0 \)

So

\[
\vec{B}_{rad} = -\mathbf{\hat{n}} \times \frac{1}{c} \frac{\mathbf{\dot{A}}_{rad}}{\partial t}
\]

\[
= \frac{1}{4\pi r} \mathbf{\hat{n}} \times \bar{m} \frac{\mathbf{\dot{\mathbf{B}}}(t_c)}{c^2}
\]

\[
= \frac{1}{4\pi r} \left( -\frac{\bar{m}}{c^2} \right)
\]

Then the radiated power is:

\[
\frac{dP}{d\Omega} = r^2 |cE\mathbf{\hat{B}} \cdot \mathbf{\hat{n}}|^2 = \frac{\bar{m}^2}{16 \pi^2 c^3} \sin^2 \Theta
\]
So the angular distribution of power $m(t) = m_0 e^{-i\omega (t - r/c)}$ is the same as the electric case, but the polarization is reversed. This a reflection of Electric-Magnetic duality, which in this context means that the fields of the magnetic dipole are related to the magnetic dipole via the rules:

$$E\text{-dipole} \rightarrow M\text{-dipole}$$

$$\vec{p} \rightarrow \vec{m}$$

$$\vec{E} \rightarrow \vec{B}$$

$$\vec{m} \rightarrow -\vec{E}$$
Relative strengths of E1 & M1 radiation

If a system has a magnetic dipole and an electric dipole, then both contribute to the radiation.

Let's compare the size of the two:

\[ \vec{p} \sim eL \]

\[ \dot{m} \sim \frac{IA}{c} \sim eL^2 \sim eL_{typ} \left( \frac{v}{c} \right) \]

So:

\[ \dot{m} \sim \frac{v}{c} \text{ is small} \]

\[ \frac{p}{c} \]

And thus the radiated power is smaller for a magnetic dipole by \((\frac{v}{c})^2\)

\[ \frac{P^{M1}}{P^{E1}} \propto \frac{m^2 \alpha (\frac{v}{c})^2}{p^2} \]
Quadrupole Radiation

Now let's compute Quadrupole radiation.

The potential fields $\Phi$ and $A$ are sourced by

\[
\frac{1}{2} \left( \frac{\partial^2 \Phi}{\partial t^2} \right) + \frac{r^2}{2} \frac{\partial^2 \Phi}{\partial r^2} - 2 \frac{\partial^2 \Phi}{\partial r^2} \cdot \frac{\partial}{\partial \tau} \frac{\partial \Phi}{\partial \tau} = -\frac{\partial}{\partial \tau} \oint_{\text{r}} \frac{\partial J^i}{\partial \tau} \frac{\partial J^i}{\partial \tau} \bigg|_{\text{r}}
\]

Using $\frac{\partial \Phi}{\partial \tau} = \delta^i_j$ and $\frac{\partial J^i}{\partial r} = -\frac{\partial \Phi}{\partial r}$

We have

\[
\frac{\partial \Phi}{\partial r} = \frac{1}{2} \left( \frac{\partial^2 \Phi}{\partial r^2} \right) = \frac{\partial^2 \Phi}{\partial r^2} \left( \frac{r^2}{2} - \frac{r^2}{3} \right) = \frac{\partial^2 \Phi}{\partial r^2} \left( \frac{r^2}{2} - \frac{r^2}{3} \right)
\]

So then

\[
A^i = \frac{\hat{n}_i}{\sqrt{1 - \frac{v^2}{c^2}}} \int_{r_0} \frac{\partial \Phi}{\partial r} \left( \frac{r^2}{2} - \frac{r^2}{3} \right)
\]

\[
= \frac{\hat{n}_i}{\sqrt{1 - \frac{v^2}{c^2}}} \int_{r_0} \frac{\partial \Phi}{\partial r} \left( \frac{r^2}{2} - \frac{r^2}{3} \right)
\]

\[
\equiv \frac{\hat{n}_i}{\sqrt{1 - \frac{v^2}{c^2}}} \int_{r_0} \frac{\partial \Phi}{\partial r} \left( \frac{r^2}{2} - \frac{r^2}{3} \right)
\]

\[
\frac{\partial A^i}{\partial r} = \frac{\hat{n}_i}{\sqrt{1 - \frac{v^2}{c^2}}} \int_{r_0} \frac{\partial \Phi}{\partial r} \left( \frac{r^2}{2} - \frac{r^2}{3} \right)
\]

\[
\frac{\partial A^i}{\partial r} = \frac{\hat{n}_i}{\sqrt{1 - \frac{v^2}{c^2}}} \int_{r_0} \frac{\partial \Phi}{\partial r} \left( \frac{r^2}{2} - \frac{r^2}{3} \right)
\]
Or in matrix notation

\[ \dot{A} = \frac{1}{24\pi r c^2} Q \cdot n \]

\[ \dot{A}_t = \dot{A} - \dot{n} (\dot{n} \cdot \dot{A}) \]

\[ \dot{E} = -\frac{1}{c} \frac{\partial A_t}{\partial t} = -\frac{1}{24\pi r c^3} (1 - nn^T) \cdot \dot{Q} \cdot n \]

\[ \dot{E} = \left[ \ddot{Q} \cdot n - \dot{n} (n^T \dot{Q} \cdot n) \right] \]

Now

\[ \frac{d\mathbf{p}}{d\Omega} = c \left| \mathbf{r} \times \mathbf{E} \right|^2 = \frac{1}{(24\pi)^2 c^5} \left[ \dot{Q} \cdot n - n (n^T \dot{Q} \cdot n) \right]^2 \]

Take a specific component to gain intuition

\[ Q_{ij} = \begin{pmatrix} -Q_{zz} & \frac{1}{2} & \frac{1}{2} \\ -Q_{zz} & -Q_{zz} & Q_{zz} \\ \frac{1}{2} & \frac{1}{2} & -Q_{zz} \end{pmatrix} \quad \text{only } Q_{zz} \text{ specified} \]

Then take

\[ \vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]
Find for this specific case \( [\mathbf{q} \cdot \mathbf{n} - n(n^2 \mathbf{q} \cdot \mathbf{n})] \right) = \) work it out to find
\[
\frac{dp}{d\Omega} = \frac{1}{(2\pi)^2 c^5} \left[ \frac{9 \mathbf{Q} \cdot \mathbf{Q} \sin^2(2\theta)}{16} \right]
\]

So we plot

So we see two characteristic lobes associated with quadrupole radiation.

It is possible to compute the total power is (Homework) in general:

\[
P = \int d\Omega \frac{dp}{d\Omega}
\]

\[
P = \frac{1}{720\pi c^5} \mathbf{Q}^i \mathbf{Q}^j
\]

For harmonic sources \( \mathbf{Q}(t) = Q_0 e^{-i\omega t} \) pick up \( \pm \) from average over time:

\[
P = \frac{c}{1440\pi} \left( \frac{\omega}{c} \right)^6 Q^i Q^j Q^{*i} Q^{*j}
\]

\( \Rightarrow \) one sees a characteristic \( \omega^6 \) dependence.
Comparison @ Dipole Radiation

For dipole radiation, \( p \approx eL \) and

\[
P \sim c \left( \frac{\omega}{c} \right)^4 p^2
\]

Power

\[
\sim c e^2 k^2 (kL)^2
\]

\[
k = \frac{\omega}{c} = \frac{2\pi}{\lambda}
\]

While for quadrupole radiation, the power is

\[
P \sim c \left( \frac{\omega}{c} \right)^6 Q_0^2, \text{ where } Q_0 \sim eL^2
\]

So

\[
P \sim c e^2 k^2 (kL)^4
\]

So units check:

\[
\frac{\text{velocity} \times \text{force}}{c e^2 k^2} = \text{Energy / time}
\]

So we see that quadrupole radiation is suppressed relative to (electric)dipole radiation by \( (kL)^2 \), i.e.,

\[
\left( \frac{L}{\lambda_{\text{typ}}} \right)^2
\]