Last Time

We introduced a whole new notation for all of electrodynamics. The reason we did this is because the notation tells us how to find the fields/momenta in another frame given measurements in others.

Rules:

1. Tensor indices go up and down with the metric tensor

\[ V^{\nu}_\mu = g_{\rho\sigma} g^{\sigma\nu} V^{\rho\mu} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

This just changes the sign of the zero component, e.g., \( P^0 = (E/c, \mathbf{p}) \) and \( P_\mu = (-E/c, \mathbf{p}) \). \( F^{0i} = E^i = -F^i \).

2. Tensor indices are transformed with the Lorentz Transform:

\[ V'^{\mu\nu} = L^\nu_\sigma L^\mu_\rho V^{\rho\sigma} \]

Lower indices transform as inverse and row:

\[ V_\mu = V^\nu (L^{-1})^\nu_\mu \quad \text{or} \quad V_\mu = (L^{-1})^\nu_\mu V^\nu \]

row view \quad \text{column view}
Then we said that $g_{\mu\nu}$ is an invariant tensor or that

$$L^\nu_\mu \equiv g_{\mu\rho} L^\rho g^\sigma_{\nu} = (L^{-1})^\nu_\mu$$

So

$$V^\nu_\mu = L^\nu_\mu V^\nu$$

Covariant Maxwell Eqs.

1. Force Law:
   - $P^\mu = (E^\mu, P^\mu)$ Invariant $P^\mu P_\mu = -(mc)^2$
   - $U^\mu = \frac{dx^\mu}{dt} = (\delta^\mu_0, \delta^\mu_i)$ with $\delta_\rho = \frac{1}{\sqrt{1 - (U_\rho/c)^2}}$

   $\equiv$ distance per proper time

   $U^\mu U_\mu = -c^2$

   $d\tau = $ proper time (time in rest frame of particle)

   $\frac{dt}{\delta_\rho} = d\tau \quad dx^\mu = (ct, \vec{x})$

   $d\tau = dx^\mu = -c^2 d\tau^2$
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• Field strength = $F^\mu{}\nu$  \[ F^\mu{}\nu = \gamma^\mu A^\nu - \gamma^\nu A^\mu \]

\[
F^\mu{}\nu = \begin{pmatrix}
0 & E^x & E^y & E^z \\
-E^x & 0 & B^z & -B^y \\
-E^y & -B^z & 0 & B^x \\
-E^z & B^y & -B^x & 0 \\
\end{pmatrix}
\]

So  \[ E^i = F^0{}^i = -F^{i0} = F_i \]

\[ \varepsilon^{ijk} B_k = F^{ij} = -F^{ji} \]

• So with that Find

\[
\frac{dp^\mu}{dt} = q F^\mu{}\nu \frac{u^\nu}{c} \leftrightarrow \frac{dE}{dt} = q \vec{E} \cdot \vec{v} \\
\frac{d\vec{p}}{dt} = q (\vec{E} + \vec{v} \times \vec{B}) \frac{1}{c}
\]

2 Field Equations:

\[-\partial^\mu F^\mu{}^\nu = J^\nu \quad J^\nu = (e \rho, \vec{j}) \]

or equivalently

\[ \nabla \cdot \vec{E} = \rho \]

\[-\frac{1}{c} \partial_t \vec{E} + \nabla \times \vec{B} = \vec{J}/c \]
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(3) The remaining equations

\[ \nabla \cdot B = 0 \]
\[- \nabla \times B - \nabla \times E = 0 \]

Same as first two with \( E \to B \) and \( B \to -E \)

\[
F^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} \hat{F}^\alpha \beta = \begin{pmatrix}
0 & B^x & B^y & B^z \\
-B^x & 0 & -E^z + E^y & -E^x \\
-B^y & E^z & 0 & -E^x \\
B^z & -E^y & E^x & 0
\end{pmatrix}
\]
dual field

Leading to

\[-\partial F^{\mu \nu} = 0\]

or equivalently (in terms of \( F_{\mu \nu} \))

\[
\partial [F_{\mu_1 \mu_2 \mu_3}] = 0
\]
denotes antisymmetric part

\[
\partial_{\mu_1} F_{\mu_2 \mu_3} - \partial_{\mu_2} F_{\mu_1 \mu_3} + \partial_{\mu_3} F_{\mu_1 \mu_2} = 0 \quad \text{order of indices like determinant}
\]

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 3 & 3 & 3
\end{vmatrix}
\]
Short in Class Exercise - Invariants of $F_{\mu \nu}$

Given $F_{\mu \nu}$ and $\tilde{F}_{\mu \nu}$ the only two invariants quadratic in the fields are

$$F_{\mu \nu} F^{\mu \nu} \text{ and } F_{\mu \nu} \tilde{F}^{\mu \nu}$$

Evaluate these in terms of $E$ and $B$:

Ans:

$$F_{\mu \nu} F^{\mu \nu} = 2(B^2 - E^2) = 2 F_{0i} F^{0i} + F_{ij} F^{ij}$$

$$F_{\mu \nu} \tilde{F}^{\mu \nu} = -4 B \cdot E$$

Solution:

$$\tilde{F}^{\mu \nu} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & E^x & 0 \end{pmatrix}$$

$$F^{\mu \nu} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ +E^x & 0 & B^z & -B^y \\ E^y & -B^z & 0 & B^x \\ E^z & B^y & -B^x & 0 \end{pmatrix}$$

Just multiply it out
Transformation of Fields

\[ F_{\mu \nu}(x) = L_\alpha L_\beta F_{\alpha \beta} \]

For a boost in the \( x \)-direction

\[ L_{\mu \nu} = \begin{pmatrix} \gamma & -\gamma \beta \\ \gamma \beta & \gamma \end{pmatrix} \]

Then first look in parallel \( x \) direction

\[ E_x = E^x = F^{01} = L^0 L_1 F_{\alpha \beta} \]
\[ = L^0 L_1 F^{01} + L^0 L^0 F^{10} \]
\[ = (\gamma^2 - \gamma^2 \beta^2) F^{01} \]
\[ = E_x' \]

Similarly in \( y \)-direction

\[ E_y = E^y = F^{02} = L^0 L_2 F_{\alpha \beta} \]
\[ = L^0 L_2 F^{02} + L^0 L^2 F^{10} \]
\[ E^2 = \gamma E_x^2 + \gamma \beta B^3 \]
\[
\begin{align*}
\vec{E}'' &= \vec{E}'' \\
\vec{B}'' &= \vec{B}'' \\
\vec{E}_1 &= \vec{E}_1 + \vec{\beta} \times \vec{B}_1 \\
\vec{B}_1 &= \vec{E}_1 - \vec{\beta} \times \vec{E}_1
\end{align*}
\]

1) Looks like coordinate transforms, but it is the transverse pieces which get boosted.

2) The transformation of \( \vec{B} \) is the dual of \( \vec{E} \):
\[
\vec{E} \rightarrow \vec{B}, \quad \vec{B} \rightarrow -\vec{E}
\]

3) Very often one has an electrostatic field \( \vec{E} = E_0 \hat{z}, \quad \vec{B} = 0 \) and we want to know \( \vec{B} \) in the new frame:
\[
\begin{align*}
\vec{B} &= -\vec{\beta} \times \gamma \vec{E}_1 \\
&= -\vec{\beta} \times \vec{E}_1 \\
&= -\vec{\beta} \times \vec{E}_1 \\
\text{components } \vec{\beta} \cdot \vec{B} &= 0 \\
\end{align*}
\]

so we can drop \( \perp \) symbol.
Last Time

- Discussed Transformation Rules for $\tilde{E}$ and $\tilde{B}$

Covariantly:

$$F^{\mu\nu}(x) = L^{\mu\rho} L^{\nu}_{\sigma} F^{\rho\sigma}(x)$$

Non-covariantly (see picture below)

$$\tilde{E}'' = \tilde{E}'' \quad \tilde{B}'' = \tilde{B}''$$

$$\tilde{E}_1 = \gamma\tilde{E}_1 + \gamma\tilde{B} \times \tilde{B}_1 \quad \tilde{B}_1 = \gamma\tilde{B}_1 - \gamma\tilde{B} \times \tilde{E}_1$$

The Invariants are

$$F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = -4 B \cdot E$$

\[\text{Diagram:} \quad E, B = \text{something} \quad E = ? \quad \begin{array}{c}
\end{array}\]

\[\text{Diagram:} \quad B = ? \quad \begin{array}{c}
\end{array}\]
Last Time

- Discussed Transformation Rules for $\vec{E} + \vec{B}$

  **Covariantly:**

  $$ F^{mn}(x) = L^m_\rho L^n_\sigma F^{\rho\sigma}(x) $$

  **Non-covariantly (see picture below):**

  $$ E'' = E'' \quad B'' = B'' $$

  $$ E_\perp = \gamma E_\perp + \gamma\beta \times B_\perp$$  \hspace{1cm}  $$ B_\perp = \gamma B_\perp - \gamma\beta \times E_\perp$$

  **The Invariants are:**

  $$ F_{mn} F^{mn} = 2(B^2 - E^2) $$

  $$ F_{mn} \tilde{F}^{mn} = -4B \cdot E $$
Fields of a moving particle

- Particle at Rest:

\[ F^{\mu\nu} = \text{Coulomb field} \]

\[ F^{\mu\nu}(x) = ? \]

- Person sees a particle approach:

- The boost to the person frame from the particle frame is:

\[ L^{\mu\nu} = \begin{pmatrix} \gamma + \gamma \beta \\ + \gamma \beta \gamma \\ + \gamma \beta \gamma \end{pmatrix} \]

\[ \hat{\beta} = \text{velocity of particle as seen by person} = -\hat{\beta}_{\text{frame}} \]
Now

\[ E_{1} (x) = \frac{q}{4\pi(x \cdot x^{'})^{3/2}} \quad x_{1} = E_{11} (x) \Rightarrow \frac{\dot{u}}{u} = \frac{q}{4\pi r^{3}} \]

\[ E_{2} (x) = \frac{q}{4\pi(x \cdot x^{'})^{3/2}} \quad x_{2} = (E_{1})^{2} \Rightarrow \frac{\dot{b}}{b} = \frac{q}{4\pi r^{3}} \]

So under boost:

\[ x^{\mu} = L^{\mu}_{\nu} x^{\nu} \]

\[ (L^{-1})^{\nu}_{\mu} x^{\mu} = x^{\nu} \]

\[ (x^{0}) = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x^{1} \end{pmatrix} = \begin{pmatrix} \frac{ct}{\sqrt{\gamma}} \\ \frac{x^{1}}{\sqrt{\gamma}} \end{pmatrix} \]

\[ x^{1} = \gamma (x^{1} - \gamma t) \]

\[ x^{2} = \frac{x_{2}}{\gamma} \]

\[ x^{3} = \frac{x_{3}}{\gamma} \quad b = (x_{2}, x_{3}) \]

So using our rules:

\[ E_{11} (x) = E_{11} (x) \]

\[ E_{11} = \frac{q \gamma (x^{1} - \gamma t)}{4\pi} \]

\[ \frac{1}{\gamma^{2}(x^{1} - \gamma t)^{2} + b^{2}}^{3/2} \]

\[ E_{11} (x) = \gamma E_{\perp} (x) = \frac{q \gamma b}{4\pi} \]

\[ \frac{1}{\gamma^{2}(x^{1} - \gamma t)^{2} + b^{2}}^{3/2} \]

\[ \vec{B} = \vec{\beta} \vec{E} \quad \text{note sign is opposite because } \vec{\beta} \text{ is velocity of particle not the frame} \]
So the picture at large $\gamma$ is

$$\Delta x = (x^i - vt) \sim \frac{b}{\gamma}$$

(Side View)

$\downarrow b$

$\downarrow v$

$\leftarrow vt$

$\rightarrow$ Person

Comment:

- The field is appreciable when $x^i - vt$ is of order $b/\gamma$

- Plotting the field strength one sees that the transverse field is much larger

$$E_{\parallel}^{\max} \sim \frac{q}{4\pi b^2}$$

$$E_{\perp} = \frac{q}{4\pi b^2}$$

i.e. the transverse fields are enhanced by $\gamma$ while the longitudinal fields remain finite
Find that the transverse fields give a finite impulse, but the longitudinal kick is zero (Homework).

Before

\[
\begin{align*}
\vec{E} & \sim \vec{B} \\
\vec{E} \times \vec{B} &= \hat{z}
\end{align*}
\]

After

\[
\begin{align*}
\Delta p_\perp &= \frac{q^2}{2\pi\hbar c} \\
\vec{V}
\end{align*}
\]

- For \( \beta \rightarrow 1 \) the \( \vec{E} \) and \( \vec{B} \) fields act like plane waves, i.e.

\[
\begin{align*}
\vec{E} & \sim \vec{B} \\
\vec{E} \times \vec{B} &= \hat{z}
\end{align*}
\]

The head on view, \( \vec{B} \) is perpendicular to \( \vec{E} \)

\[
\hat{z} = c (\vec{E} \times \vec{B}) \text{ points out of the page}
\]
The fact that \( \mathbf{B} \) is perpendicular to \( \mathbf{E} \) could have been anticipated

\[ F_{\mu\nu} F^{\mu\nu} = -4 E \cdot B \]

is Lorentz invariant. In the frame of the particle, \( \mathbf{B} \) is zero so in the particle frame:

\[ F_{\mu\nu} \cdot F^{\mu\nu} = 0 = \]

So in any other frame we must have \( E \cdot B = 0 \)
Moving Media

- For stationary conductors

\[ \mathbf{j} = \sigma \mathbf{E} + \mathbf{c} \mathbf{\times} \mathbf{B} \]

- It is reasonably clear that the appropriate generalization is for moving conductors with velocity \( \mathbf{v} \) is:

\[ \mathbf{j} = \sigma (\mathbf{E} + \frac{\mathbf{v}}{c} \mathbf{\times} \mathbf{B}) \]

to first order in \( \frac{v}{c} \). Let's see it

\[ \begin{array}{c}
\text{Sample} \\
E, B \\
O \\
\mathbf{V}
\end{array} \]

Electric field in sample frame

Current in sample frame

Sample \( \mathbf{j} = \sigma \mathbf{E} \approx \sigma (\mathbf{E} + \frac{v}{c} \mathbf{\times} \mathbf{B}) \)

from transformation

Now we have properties of \( \mathbf{E} + \mathbf{B} \)

to compute the current in lab frame, no net charge

\[ \begin{pmatrix} \mathbf{\dot{E}} \\ \mathbf{\dot{B}} \end{pmatrix} = \begin{pmatrix} \gamma + \gamma' \beta & 0 \\ 0 & \gamma + \gamma' \beta \end{pmatrix} \begin{pmatrix} \mathbf{\dot{E}} \\ \mathbf{\dot{B}} \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{\dot{B}} \end{pmatrix} \]

Current in lab

So

\[ \mathbf{j}_{\text{Lab}} \approx \mathbf{j} \approx \sigma (\mathbf{E} + \frac{v}{c} \mathbf{\times} \mathbf{B}) \]
Example

A uniform conducting cylinder rotated with angular velocity \( \omega \), determine the torque required to maintain the motion. Cylinder has conductivity \( \sigma \) and radius \( a \).

Solution

\[
\frac{J}{\sigma} = \mathbf{V} \times \mathbf{B} = \frac{-\sigma \omega r (-\sin \phi \mathbf{\hat{x}} + \cos \phi \mathbf{\hat{y}})}{c} \times \mathbf{B}_0 \mathbf{\hat{x}}
\]

\[
\frac{J}{\sigma} = \mathbf{B}_0 \frac{\sigma \omega r \cos \phi}{c} (-\mathbf{\hat{z}})
\]

So the forces are

The remaining steps are easy:

\[
\mathbf{T}_{em} = L \int d^2 \tau \quad \mathbf{F} \times \left( \frac{J}{\sigma} \mathbf{\hat{B}} \right) = \int d^2 \tau \quad \frac{J}{\sigma} \left( \mathbf{F} \cdot \mathbf{\hat{B}} \right) - \left( \frac{J}{\sigma} \mathbf{\hat{B}} \right) \mathbf{F}
\]

\[
= L \int_0^a d \rho \int_0^{2\pi} dr d\phi \quad (-\mathbf{\hat{z}}) \left( \mathbf{B}_0 \sigma \omega r \cos \phi / c^2 \right) \left( \mathbf{B}_0 \gamma \cos \phi \right)
\]
So then,

\[ \frac{T_{em}}{L} = \text{torque per length} \]

\[ = \left( \frac{B_0^2 a^4 \sigma \omega}{c^2} \right) (-k) \]

Units:

\[ B_0^2 = \text{N/m}^2 \]

\[ a^4 = m^4 \]

\[ \omega \sigma = 1 \]

\[ \left[ \frac{T_{em}}{L} \right] = \left[ \frac{N \ m^4 \ s^{-1}}{m^2 \ s^2 \ s^2} \right] = N \ s \]

\[ \sigma = \frac{m^2}{s^2} \]

\[ C^2 = \frac{m^2}{s^2} \]