Last Time

We introduced a whole new notation for all of electrodynamics. The reason we did this is, is because the notation tells us how to find the fields/momenta in another frame given measurements in others.

Rules:

① Tensor indices go up and down with the metric tensor

$$V^{\alpha}{}_{\beta\gamma} = g_{\beta\mu} g_{\gamma\nu} V^{\alpha\mu\nu} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

This just changes the sign of the zero components, example, $P^{\mu} = \left(\frac{E}{c}, \vec{p}\right)$ and $P_{\mu} = \left(-\frac{E}{c}, p\right)$, $F^{0i} = E^{i} = -F_{0}{}^{i}$

② Tensor indices are transformed with the Lorentz Transform:

$$V^{\mu\nu} = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} V^{\rho\sigma}$$

Lower indices transform as inverse and row:

$$\underbrace{V_{\mu} = V_{\nu} (L^{-1})^{\nu}{}_{\mu}}_{\text{row view}} \quad \text{or} \quad \underbrace{V_{\mu} = (L^{-1T})_{\mu}{}^{\nu} V_{\nu}}_{\text{column view}}$$

Then we said that $g_{\mu\nu}$ is an invariant tensor or that

$$L_\mu{}^\nu \equiv g_{\mu\rho} L^\rho{}_\sigma g^{\sigma\nu} = (L^{-1T})_\mu{}^\nu$$

So

$$V_\mu = L_\mu{}^\sigma V_\sigma$$

Covariant Maxwell Eqs:

① Force Law:

- $P^\mu \equiv \left(\frac{\mathcal{E}}{c}, \vec{P}\right)$: Invariant $P^\mu P_\mu = -(mc)^2$

- $U^\mu \equiv \frac{dx^\mu}{d\tau} = (\gamma_p c, \gamma_p \vec{v}_p)$ with $\gamma_p = \frac{1}{\sqrt{1-(v_p/c)^2}}$

$\equiv$ distance per propper time

$$U_\mu U^\mu = -c^2$$

- $d\tau \equiv$ propper time (time in rest frame of particle)

$$\frac{dt}{\gamma_p} = d\tau \qquad dx^\mu = (ct, \vec{x})$$

$$dx_\mu dx^\mu = -c^2 d\tau^2$$

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- Field Strength $= F^{\mu\nu}$ $\quad\quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$F^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & \vec{E}^x & E^y & E^z \\ \hline -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{array}\right)$$

So $\quad E^i = F^{0i} = -F^{i0} = F^i{}_0$

$$\varepsilon^{ijk} B_k = F^{ij} = -F^{ji}$$

- So with that Find

$$\boxed{\frac{dP^\mu}{d\tau} = q F^\mu{}_\nu \frac{u^\nu}{c}} \iff \boxed{\begin{aligned} dE/dt &= q \vec{E} \cdot \vec{v} \\ d\vec{p}/dt &= q\left(E + \frac{\vec{v} \times \vec{B}}{c}\right) \end{aligned}}$$

② Field Equations:

$$-\partial_\mu F^{\mu\nu} = \frac{J^\nu}{c} \quad\quad J^\nu = (c\rho, \vec{J})$$

or equivalently

$$\nabla \cdot E = \rho$$

$$-\frac{1}{c}\partial_t E + \nabla \times B = J/c$$

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3 The remaing equations

$$\nabla \cdot B = 0$$

$$-\frac{1}{c}\partial_t B - \nabla \times E = 0$$

Same as first two with $E \to B$ and $\vec{B} \to -\vec{E}$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & +E^y \\ -B^y & E^z & 0 & -E^x \\ B^z & -E^y & E^x & 0 \end{pmatrix}$$

dual field

Leanding to

$$-\partial_\mu \tilde{F}^{\mu\nu} = 0$$

or equivalently (in terms of $F_{\mu\nu}$)

$$\partial_{[\mu_1} F_{\mu_2\mu_3]} = 0$$

denotes antisymmetric part

$$\partial_{\mu_1} F_{\mu_2\mu_3} - \partial_{\mu_2} F_{\mu_1\mu_3} + \partial_{\mu_3} F_{\mu_1\mu_2} = 0$$

order of indices like determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix}$$

Short in Class Excercise - Invariants of $F^{\mu\nu}$

Given $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ the only two invariants quadratic in the fields are

$$F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Evaluate these in terms of E and B:

Ans:

$$F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2) = \overbrace{2F_{0i}F^{0i}}^{-2E^2} + \overbrace{F_{ij}F^{ij}}^{2B^2}$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\, B\cdot E$$

Solution:

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & E^z & -E^y \\ -B^y & -E^z & 0 & E^x \\ -B^z & E^y & -E^x & 0 \end{pmatrix} \qquad F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & -E^x & -E^y & -E^z \\ \hline +E^x & 0 & B^z & -B^y \\ E^y & -B^z & 0 & B^x \\ E^z & B^y & -B^x & 0 \end{array}\right)$$

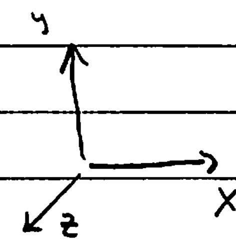
Just multiply it out

Transformation of Fields

$$\underline{F}^{\mu\nu}(x) = L^{\mu}{}_{\alpha}\, L^{\nu}{}_{\beta}\, F^{\alpha\beta}$$

For a boost in the x-direction

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$



- Then first look in parallel x_1 direction

$$\underline{E}_{\parallel} = \underline{E}^{x} = \underline{F}^{01} = L^{0}{}_{\alpha} L^{1}{}_{\beta} F^{\alpha\beta}$$

$$= L^{0}{}_{0} L^{1}{}_{1} F^{01} + L^{0}{}_{1} L^{1}{}_{0} F^{10}$$

$$= (\gamma^2 - \gamma^2\beta^2)\, F^{01}$$

$$= E_{\parallel}$$

- Similarly in $\perp$ direction

$$\underline{E}_{\perp} = \underline{E}^{y} = \underline{F}^{02} = L^{0}{}_{\alpha} L^{2}{}_{\beta} F^{\alpha\beta}$$

$$= L^{0}{}_{0}\, L^{2}{}_{2}\, F^{02} + L^{0}{}_{1}\, L^{2}{}_{2}\, F^{12}$$

$$\underline{E}^{2} = \gamma E^{2} + \gamma\beta B^{3}$$

So we find

$$\boxed{\begin{aligned} \underline{\vec{E}}^{\parallel} &= \vec{E}^{\parallel} & \quad \underline{\vec{B}}^{\parallel} &= \vec{B}^{\parallel} \\ \underline{\vec{E}}_{\perp} &= \gamma \vec{E}_{\perp} + \gamma \vec{\beta} \times \vec{B}_{\perp} & \quad \underline{\vec{B}}_{\perp} &= \gamma \vec{B}_{\perp} - \gamma \vec{\beta} \times \vec{E}_{\perp} \end{aligned}}$$

① Looks like coordinate transforms, but it is the transverse pieces which get boosted.

② The transformation of $\vec{B}$ is the dual of $\vec{E}$

$\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$

③ Very often one has an electrostatic field ($\vec{E} = \vec{E}^{(0)}$, $\vec{B} = 0$), and we want to know $\underline{\vec{B}}$ in the new frame:

$$\begin{aligned} \underline{\vec{B}} &= -\vec{\beta} \times \gamma E_{\perp} \\ &= -\vec{\beta} \times \underline{E}_{\perp} \\ &= -\vec{\beta} \times \underline{\vec{E}} \end{aligned}$$

$\underline{\vec{E}}_{\perp} = \gamma E_{\perp}$

$\parallel$ components $\times \vec{\beta} = 0$ so we can drop ~~$\perp$ symbol~~

Last Time

- Discussed Transformation Rules for $\vec{E}$ & $\vec{B}$

Covariantly:

$$\underline{F}^{\mu\nu}(\underline{x}) = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} F^{\rho\sigma}(x)$$

Non-covariantly (see picture below)

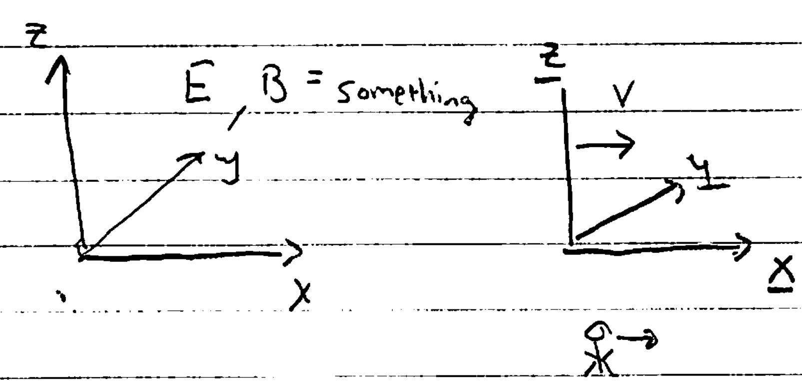
$$\underline{E}_{\parallel} = E_{\parallel} \qquad\qquad \underline{B}_{\parallel} = B_{\parallel}$$

$$\underline{E}_{\perp} = \gamma \vec{E}_{\perp} + \gamma \vec{\beta} \times \vec{B}_{\perp} \qquad \underline{\vec{B}}_{\perp} = \gamma B_{\perp} - \gamma \vec{\beta} \times \vec{E}_{\perp}$$

The Invariants are

$$F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = -4 B \cdot E$$



$\underline{E} = ?$

$\underline{B} = ?$

Last Time

- Discussed Transformation Rules for $\vec{E}$ & $\vec{B}$

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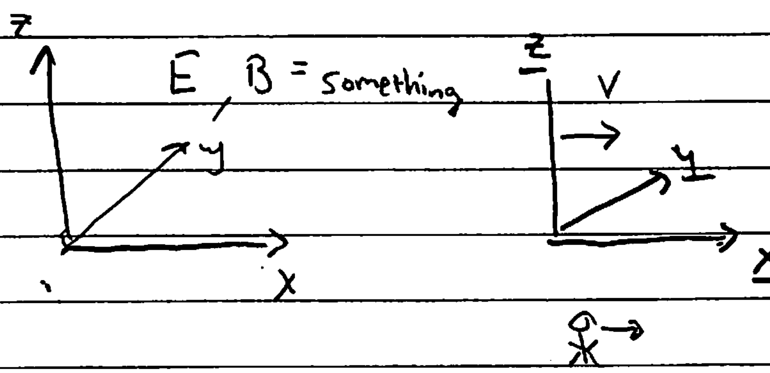


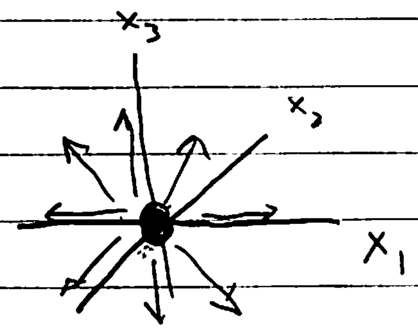
$\underline{E} = ?$

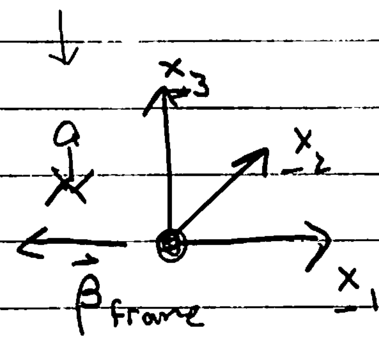
$\underline{B} = ?$

Fields of A moving Particle

- Particle at Rest:

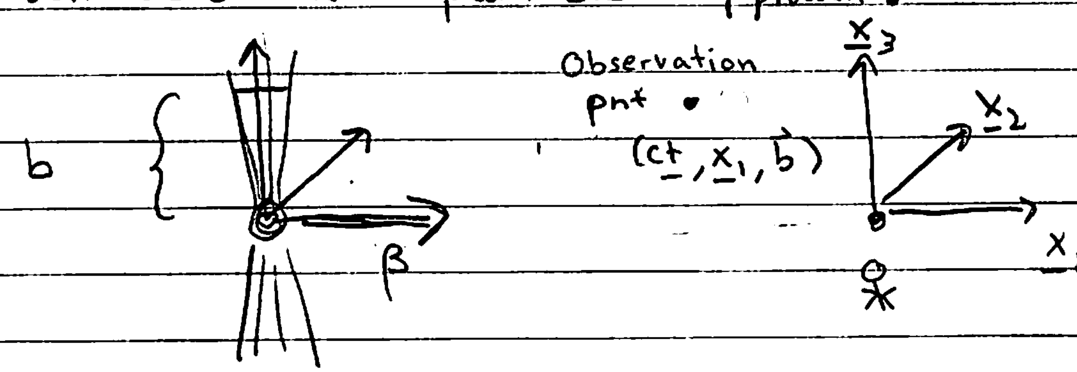
person approaching





$F^{\mu\nu}$ = coulomb field $\qquad\qquad$ $F^{\mu\nu}(x) = ?$

- Person sees a particle approach:



- The boost to the person frame from the partice frame is:

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & +\gamma\beta & & \\ +\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\vec{\beta}$ = velocity of particle as seen by person
$= -\vec{\beta}_{frame}$

Now

$$E_1(x) = \frac{q}{4\pi(x_i x^i)^{3/2}} x_1 = E_{\parallel}(x) \Rightarrow \vec{E}_{\parallel} = \frac{q\,\vec{z}}{4\pi r^3}$$

$$E_2(x) = \frac{q}{4\pi(x_i x^i)^{3/2}} x_2 = (\vec{E}_{\perp})_2 \Rightarrow \vec{E}_{\perp} = \frac{q\,\vec{b}}{4\pi r^3}$$

So under boost:

$$\left.\begin{aligned} \underline{x}^{\mu} &= L^{\mu}{}_{\nu}\, x^{\nu} \\ (L^{-1})^{\nu}{}_{\mu}\, \underline{x}^{\mu} &= x^{\nu} \end{aligned}\right\} \qquad \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & \\ -\gamma\beta & \gamma & \\ & & 1 \end{pmatrix} \begin{pmatrix} c\underline{t} \\ \underline{x}^1 \\ \underline{b} \end{pmatrix}$$

$$x^1 = \gamma(\underline{x}^1 - v\underline{t})$$

$$\begin{matrix} x^2 \\ x^3 \end{matrix} = \begin{matrix} \underline{x}_2 \\ \underline{x}_3 \end{matrix} \qquad \vec{b} = (x_2, x_3)$$

So using our rules:

$$\underline{E}_{\parallel}(\underline{x}) = E_{\parallel}(x)$$

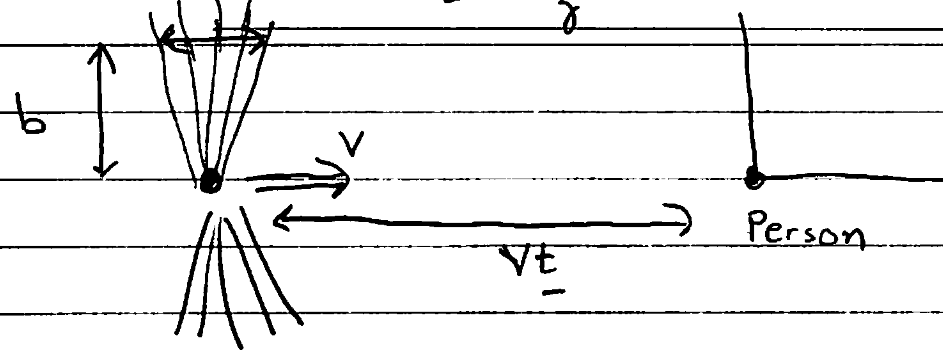
$$\underline{E}_{\parallel} = \frac{q\gamma(\underline{x}^1 - v\underline{t})/4\pi}{[\gamma^2(\underline{x}^1 - v\underline{t})^2 + b^2]^{3/2}}$$

$$\underline{E}_{\perp}(\underline{x}) = \gamma E_{\perp}(x) = \frac{q\gamma\,\vec{b}/4\pi}{[\gamma^2(\underline{x}^1 - v\underline{t})^2 + b^2]^{3/2}}$$

$\vec{\underline{B}} = \vec{\beta} \times \vec{\underline{E}}$ ← note sign is opposite because $\vec{\beta}$ is velocity of particle not the frame $= \vec{\beta}_{frame}$

So the picture at Large γ is

$\Delta x = (\underline{x}' - v\underline{t}) \sim \frac{b}{\gamma}$



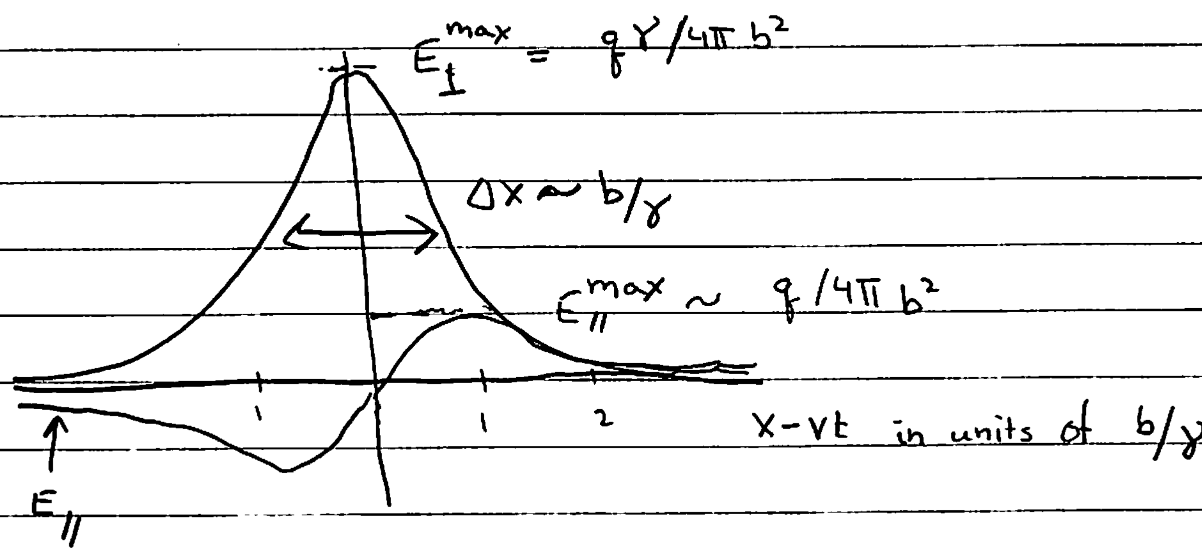
(Side View)

Comments

longitudinal dist from particl

- The field is appreciable when $\underline{x}_1 - v\underline{t}$ is of order b/γ

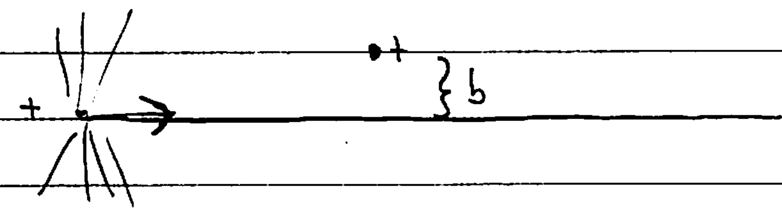
- Plotting the field strengths one sees that the transverse field is much larger



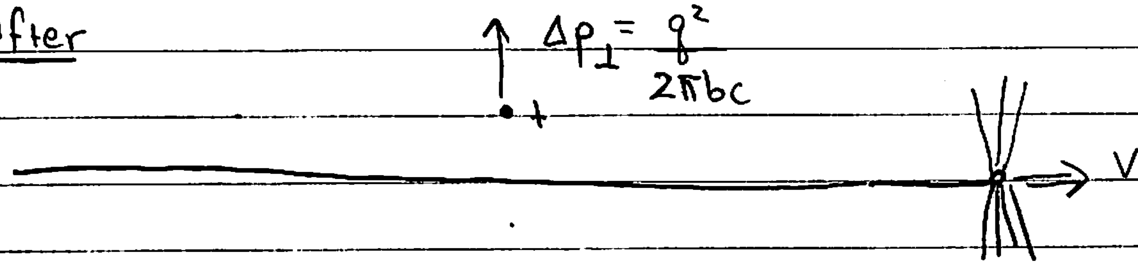
i.e. the transverse fields are enhanced by γ while the longitudinal fields remain finite

Find that the transverse fields give a finite impulse, but the longitudinal kick is zero (Homework).

Before



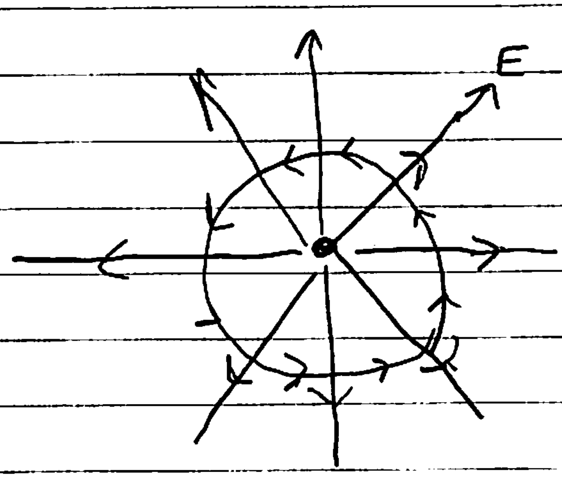
After



- For $\beta \to 1$ the $\vec{E}$ and $\vec{B}$ fields act like plane waves, ie

$$|\vec{E}| \simeq |\vec{B}| \qquad \hat{E} \times \hat{B} = \hat{z}$$

The head on view, $\vec{B}$ is perpendicular to $\vec{E}$



$\vec{S} = c\,(\vec{E} \times \vec{B})$ points out of the page

• The fact that $\vec{B}$ is perpendicular to $\vec{E}$ could have been anticipated

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B}$$

is Lorentz invariant. In the frame of the particle, $\vec{B}$ is zero so in the particle frame:

$$F_{\mu\nu}\cdot\tilde{F}^{\mu\nu} = 0 \quad =$$

So in any other frame we must have $\vec{E}\cdot\vec{B} = 0$

Moving Media

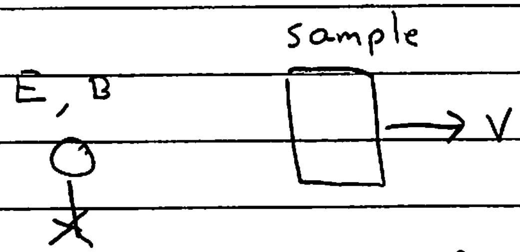
- For stationary conductors

$$\vec{j} = \sigma \vec{E} + \overset{\text{small}}{\cancel{c\chi^B_m \nabla \times B}}$$

- It is reasonably clear that the appropriate generalization is for moving conductors with velocity $\vec{V}$ is:

$$\vec{j} = \sigma \left(E + \frac{\vec{V}}{c} \times \vec{B}\right)$$

to first order in $\vec{V}/c$. Lets see it



current in sample frame

electric field in sample frame

$$\vec{j} = \sigma \underline{E} \simeq \sigma \left(E + \frac{v}{c} \times B\right)$$

from transformation properties of E & B

Now we have to compute the current in lab frame

current in lab

no net charge in sample

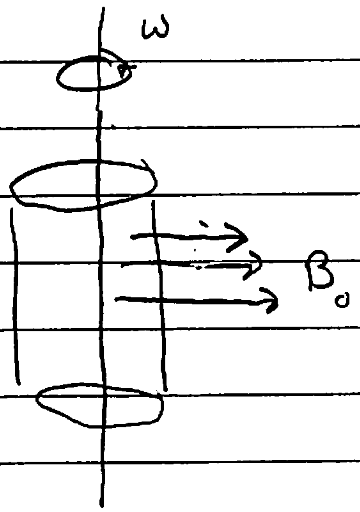
$$\begin{pmatrix} c\rho \\ j \end{pmatrix}_{Lab} = \begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix} \simeq \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vec{j} \end{pmatrix}$$

sample current

So

$$\boxed{\vec{j}_{Lab} \simeq \underline{\vec{j}} \simeq \sigma\left(E + V/c \times B\right)}$$

Example



$$\vec{B} = B_0 \hat{x}$$

A uniform conducting cylinder rotated with angular velocity ω. Determine the torque required to maintain the motion. Cylinder has conductivity σ and radius a

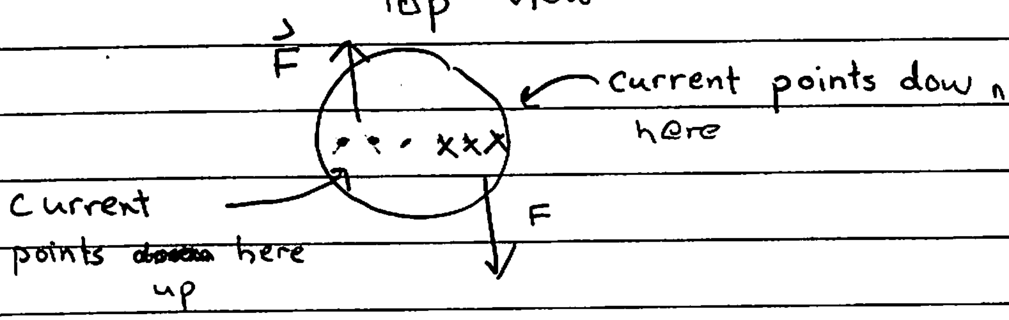
Solution

$$\vec{j} = \sigma \frac{\vec{v}}{c} \times \vec{B} = \sigma \overbrace{\omega r \left(-\sin\phi\, \hat{x} + \cos\phi\, \hat{y}\right)}^{\vec{v}} \times B_0 \hat{x}$$

$$\vec{j} = \frac{B_0}{c} \sigma \omega r \cos\phi \,(-\hat{z})$$

So the forces are

Top view



The remaining steps are easy:

$$\vec{\tau}_{em} = \overset{\text{length}}{L} \int d^2 r \;\; \vec{r} \times \left(\frac{\vec{j}}{c} \times \vec{B}\right) = \int d^2\vec{r} \;\; \frac{\vec{j}}{c}\,(\vec{r} \cdot \vec{B}) - \overset{0}{\cancel{\left(\vec{r} \cdot \frac{\vec{j}}{c}\right)}} \vec{B}$$

$$= L \int_0^a r\, dr \int_0^{2\pi} d\phi \;\; (-\hat{z}) \left(B_0 \sigma \omega r \cos\phi / c^2\right) \left(B_0 r \cos\phi\right)$$

So then

$$\frac{\vec{\tau}_{em}}{L} = \text{torque per length}$$

$$= \left(B_0^2 \frac{a^4}{4} \frac{\sigma \omega}{c^2} \cdot \pi \right) (-\hat{z})$$

Units:

$B_0^2 = N/m^2$

$a^4 = m^4$

$\omega \sigma = \frac{1}{s^2}$

$c^2 = \frac{m^2}{s^2}$

So

$$\left[\frac{\tau_{em}}{L} \right] = \frac{N}{m^2} \; m^4 \; \frac{1}{s^2} \; \frac{1}{\frac{m^2}{s^2}} = N \checkmark$$