Dimensional Analysis of Maxwell Eqs

\[ \nabla \cdot E = \rho \]

\[ \nabla \times B = \frac{j}{c} + \frac{1}{c} \frac{dE}{dt} \]

\[ \nabla \cdot B = 0 \]

\[ \nabla \times E = -\frac{1}{c} \frac{dB}{dt} \]

The speed of light is fast compared to macroscopic scales. Terms with \( \frac{V}{c} \) are smaller. Experiments have a characteristic length scale, \( L \), time scale, \( T \), and charge, \( Q \).

Formally define:

\[ \overline{t} = \frac{t}{T}, \quad \overline{x} = \frac{x}{L}, \quad \nabla = \nabla \overline{L} \]

\[ \overline{L} = \frac{1}{L} \left( \frac{\partial}{\partial \overline{x}}, \frac{\partial}{\partial \overline{y}}, \frac{\partial}{\partial \overline{z}} \right) \]

Similarly

\[ \overline{\rho} = \frac{\rho}{Q/L^3} \]

\[ \overline{j} = \frac{j}{Q/L^2T} \]
and most importantly

\[ \mathcal{E} \equiv \frac{c}{(L/r)} \geq 1 \quad \text{since} \ c \ is \ fast; \quad \mathcal{E} \sim 10^8 \quad \frac{1}{\mathcal{E}} \sim 10^{-8} \]

Then

\[ \nabla \cdot \mathcal{E} = \mathcal{E} \]

\[ \nabla \times \mathcal{B} = \frac{1}{\mathcal{E}} \mathcal{J} + \frac{1}{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial t} \]

\[ \nabla \cdot \mathcal{B} = 0 \]

\[ \nabla \times \mathcal{E} = -\frac{1}{\mathcal{E}} \frac{\partial \mathcal{B}}{\partial t} \]

And set up a series in \( \mathcal{E} \):

\[ E = E^{(0)} + E^{(1)} + E^{(2)} + \ldots \]

\[ B = B^{(0)} + B^{(1)} + B^{(2)} + \ldots \]

i.e., \( E^{(1)} \) is of order \( \frac{1}{\mathcal{E}} E^{(0)} \sim 10^{-8} E^{(0)} \);

\[ E^{(2)} \quad " \quad \frac{1}{\mathcal{E}^2} E^{(0)} \sim 10^{-16} E^{(0)} \]
Can continue in this way, and work out systematic corrections to electric and magnetic statics.

We will come to this later.
Fundamental Eqs. Goals:

\[ \nabla \cdot \mathbf{E} = \rho(x) \]
\[ \nabla \times \mathbf{E} = 0 \]

(1) Compute forces between charged objects
(2) Learn math

Since \( \nabla \times \mathbf{E} = 0 \), it can be written as a gradient of a scalar function (Helmholtz)

\[ \mathbf{E} = -\nabla \varphi \] Scalar potential

Alternatively

\[ \varphi(x_b) - \varphi(x_a) = -\int_{x_a}^{x_b} \mathbf{E} \cdot d\mathbf{l} \]

Substituting \( \mathbf{E} \) into \( \nabla \cdot \mathbf{E} = \rho(x) \) we find an equation for \( \varphi \)

\[ -\nabla^2 \varphi = \rho(x) \] Poisson equation

In the absence of charge \( \rho(x) \)

\[ -\nabla^2 \varphi = 0 \] Laplace
Boundary Conditions

Gauss Law:

\[ \oint \mathbf{E} \cdot d\mathbf{a} = Q_{\text{enc}} \]

\[ A (\mathbf{n} \cdot E_{\text{out}} - \mathbf{n} \cdot E_{\text{in}}) = Q_{\text{enc}} \]

\[ \mathbf{E}_{\text{out}}^{\perp} - \mathbf{E}_{\text{in}}^{\perp} = \mathbf{\sigma} \]

Similarly,

\[ \oint d\mathbf{a} \cdot \nabla \times \mathbf{E} = 0 \]

\[ \mathbf{E}_{\text{out}}^{\parallel} - \mathbf{E}_{\text{in}}^{\parallel} = 0 \]

or \( \nabla \times (\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}}) = 0 \)
Charged Plane

\[ E = \frac{\sigma}{\epsilon_0} \]

For capacitor:

\[ E = \sigma \]

Metal:

\[ E_{\text{out}} = E_{\text{n}} \hat{n} \]
\[ E_{\text{n}} = \sigma \]
\[ E_{\text{in}} = 0 \]
Boundary Conditions + Force/Area on a metal plate

\[ E = 0 \]

Magnify

Ask about force per area

\[ \frac{F}{A} = \sigma \left( E_{\text{out}} - E_{\text{self}} \right) \]

Charge/area

The part of the electric field produced by \( \sigma \)

\[ E_{\text{out}} = \sigma \]

\[ F = \sigma \left( \sigma - \frac{\sigma}{2} \right) \]

\[ \frac{F}{A} = \frac{\sigma^2}{2} \]

\( F \) from a wall of charge

\[ E_{\text{self}} \sigma \frac{1}{2} + \sigma \frac{1}{2} = E_{\text{self}} \]
Energy and Forces

because we should
sum over pairs

\[ W = \frac{1}{2} \sum_i \sum_j q_i q_j \frac{1}{4\pi |\vec{r}_i - \vec{r}_j|} \]

\[ q_1, q_4 \]

\[ q_2, q_3 \]

UE is the energy required
to assemble the charges.

When written in continuous form,

\[ W_E = \frac{1}{2} \int \int \frac{\rho(\vec{x}) \rho(\vec{x}')}{4\pi |\vec{x} - \vec{x}'|} \]

\[ = \frac{1}{2} \int \rho(\vec{x}) \varphi(\vec{x}) \]

Now \( \rho(\vec{x}) = -\nabla^2 \varphi(\vec{x}) \), so

\[ W_E = \frac{1}{2} \int \left[ -2\nabla \varphi(\vec{x}) \right] \varphi(\vec{x}) \]

By parts.

\[ W_E = \frac{1}{2} \int \left[ -2\nabla \varphi(\vec{x}) \right] \left[ -2\nabla \varphi(\vec{x}) \right] \]

\[ \left( E(\vec{x}) \cdot E_i(\vec{x}) \right) \]

\[ W_E = \frac{1}{2} \int \left( E(\vec{x}) \right)^2 \]
So the energy density is

\[ U_E = \frac{1}{2} E^2 \]
Force + Stress

\[ T_{ij} = \text{Force per area} = \text{stress tensor} \]

\[ = \text{Force in the } j\text{-th direction due to the area in the } i\text{th} \]

\[ n_i T_{ij} = \text{Force in } j\text{-th direction on outside by the inside} \]

In general lots of forces mechanical (pressure) in addition to Electrical.
Examples:

- Ideal Fluid \( T_{iy} = \rho g \)
  - at rest

\[
\hat{n} \quad F^x = n \cdot T^{ix} = T^{xx} = \rho
\]

- Viscous fluid:
  - Force

\[
F^x = -\eta \frac{\Delta V^x}{A_y} = T^{yx}
\]

\[\text{normal} \quad \Rightarrow \quad \text{viscous force slows down upper stream}\]

- Electricity, we will show that

\[
T_{iy} = -E_i E^i + \frac{1}{2} E^2 \delta^{iy}
\]
Stress Tensor and Conservation Laws

Take an element of solid and ask how the momentum per volume \( \frac{d}{dt} \) changes with respect to time.

We would expect \( \frac{d}{dt} \) to obey a conservation law:

\[
\frac{d}{dt} \sum_{\mu} \sigma_{\mu \nu} + \delta_{\nu} (T^{\nu\mu}) = 0
\Rightarrow \frac{d}{dt} \sigma_{\mu \nu} = -\delta_{\nu} T^{\nu\mu}
\]

If \( \frac{d}{dt} \) obeys an equation like this, then:

\[
\frac{d}{dt} \sum_{\mu} P^{\mu}_{\text{tor}} = \frac{d}{dt} \sum_{\nu} \sigma_{\mu \nu} = -\sum_{\nu} \delta_{\nu} T^{\nu\mu}
\]

The net force goes to zero if volume is to infinity.
Summary 

1. $\mathbf{f} = -\mathbf{\frac{d}{dt} \mathbf{T}}^{ij}_{g} = \text{force in } g\text{-th direction per volume}$
   \hspace{1cm} (\text{difference in force per area})

2. The total force on an object

\[
\frac{d\mathbf{P}^i_{tot}}{dt} = -\int \mathbf{n} \cdot \mathbf{T}^{ij}_{g} \, d\mathbf{a}
\]

Back to electrodynamics, the force is

\[ f^j = \rho \mathbf{E}^j = \text{force per vol} \]

It should be the divergence of something:

\[ f^j = \rho \mathbf{E}^j \] \hspace{1cm} \nabla \cdot \mathbf{E} = \rho \Rightarrow \nabla \cdot \mathbf{E} = 0

\[ f^j = (\mathbf{2} \cdot \mathbf{E})^{i} \mathbf{E}^i \] \hspace{1cm} \nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{2} \cdot \mathbf{E} \cdot \mathbf{j} - \mathbf{2} \cdot \mathbf{E} = 0

In class:

\[ \mathbf{E} = \mathbf{2} \cdot (\mathbf{E}^{i} \mathbf{E}^{i} - \frac{1}{2} \mathbf{g}^{ij} \mathbf{E}^{2}) \]

So:

\[ T^{ij}_{g} = -\mathbf{E}^{i} \mathbf{E}^{j} + \frac{1}{2} \mathbf{E}^{2} \mathbf{g}^{ij} \]
Solution to In Class Problems

1. \[ f^d = \rho E^d \] 
   \[ \nabla \cdot E = \rho \quad \nabla \times E = 0 \]
   \[ \rho \ n E^i = \rho \quad \partial_i E_j - \partial_j E_i = 0 \]

   \( f^d \) should be the divergence of something:

   \[ f^d = -2 \nabla^{ij} \rightarrow \text{what is } T^{ij} ? \]

Solution

\[ f^d = \rho E^d \]

from \[ \partial_i E^i - \partial^d E_d = 0 \]

\[ = (\partial_i E^i) E^d \]

\[ = 2_i (E^i E^d) - E^i \partial_i E^d \]

\[ = 2_i (E^i E^d) - E^i \partial^d (E^i E^d) \]

\[ = 2_i (E^i E^d) - \frac{1}{2} \partial^d (E^i E^d) \]

relabel \( E^i E_i = E^2 \)

\[ = 2_i \left( E^i E^d - \frac{1}{2} \delta^{ij} E^2 \right) \]

use \[ \partial_i \delta^{ij} = \delta^i \]

\[ = -2_i \left( -E^i E^d + \frac{1}{2} \delta^{ij} E^2 \right) \]

\[ T^{ij} \]
For a metal

\[ \hat{E} = \sigma \hat{n} \quad \text{or} \quad E^2 = \sigma \hat{n} \hat{n} \]

\[ \mathbf{E} = 0 \quad \text{metal} \]

\[ + \hat{n} : T^{iy} = \text{force on exterior by metal} \]

\[ - \hat{n} : T^{iy} = \text{force on metal by exterior} \]

\[ E^2 = \frac{E^2}{2} \]

\[ = -n_i \left( -E_i E^j + \frac{1}{2} E^2 \delta_{ij} \right) \]

\[ E^2 = \sigma \hat{n} \cdot \sigma \hat{n} \]

\[ = -n_i \left( -\sigma n_i \sigma n^j + \frac{1}{2} \sigma^2 \delta_{ij} \right) \]

\[ = \sigma^2 \]

\[ -n_i T^{iy} = \sigma^2 n_i - \frac{1}{2} \sigma^2 n^i \]

\[ -n_i T^{iy} = \frac{\sigma^2 n^i}{2} \]

agrees with before for force/area