Dimensional Analysis of Maxwell Eqs

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{j}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

The speed of Light is fast compared to macroscopic scales. Terms with $1/c$ are smaller. Experiments have a characteristic length scale, L, time scale, T, and charge, Q. Formally define:

$$\bar{t} \equiv \frac{t}{T} \qquad \bar{x} \equiv \frac{x}{L} \qquad \nabla \equiv \frac{1}{L}\bar{\nabla}$$

$$= \frac{1}{L}\left(\hat{x}\frac{\partial}{\partial \bar{x}} + \hat{y}\frac{\partial}{\partial \bar{y}} + \hat{z}\frac{\partial}{\partial \bar{z}}\right)$$

Similarly

$$\rho \equiv (Q/L^3)\,\bar{\rho}$$

$$j \equiv \left(\frac{Q}{L^2 T}\right)\bar{j}$$

and most importantly

$$\bar{c} \equiv \frac{c}{(\bar{L}/\bar{T})} \gg 1$$, since c is fast; $\bar{c} \sim 10^{8}$, $\frac{1}{\bar{c}} \sim 10^{-8}$

Then

$$\bar{\nabla} \cdot \bar{E} = \bar{\rho}$$

$$\bar{\nabla} \times \bar{B} = \frac{1}{\bar{c}} \bar{J} + \frac{1}{\bar{c}} \frac{\partial \bar{E}}{\partial \bar{t}}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{1}{\bar{c}} \frac{\partial \bar{B}}{\partial \bar{t}}$$

now we will stop writing the ba

And set up a series in $1/c$:

$$E = E^{(0)} + E^{(1)} + E^{(2)} + \dots$$

$$B = B^{(0)} + B^{(1)} + B^{(2)} + \dots$$

i.e. $E^{(1)}$ is of order $\frac{1}{\bar{c}} E^{(0)} \sim 10^{-8} E^{(0)}$;

$E^{(2)}$ " " " $\frac{1}{\bar{c}^2} E^{(0)} \sim 10^{-16} E^{(0)}$

Can continue in this way, and work out systematic corrections to electric and magneto statics.

We will come to this Later....

Electrostatics

Fundamental Eqs.

$$\left.\begin{aligned} \nabla\cdot E &= \rho(x) \\ \nabla\times E &= 0 \\ F &= qE \end{aligned}\right\}$$

Goals:

① Compute forces between charged objects

② Learn math

Since $\nabla\times\vec{E} = 0$ it can be written as a gradient of a scalar function (Helmholtz)

☆ $$\vec{E} = -\vec{\nabla}\varphi \leftarrow \text{Scalar potential}$$

Alternatively

$$\varphi(x_b) - \varphi(x_a) = -\int_a^b \vec{E}\cdot d\vec{l}$$

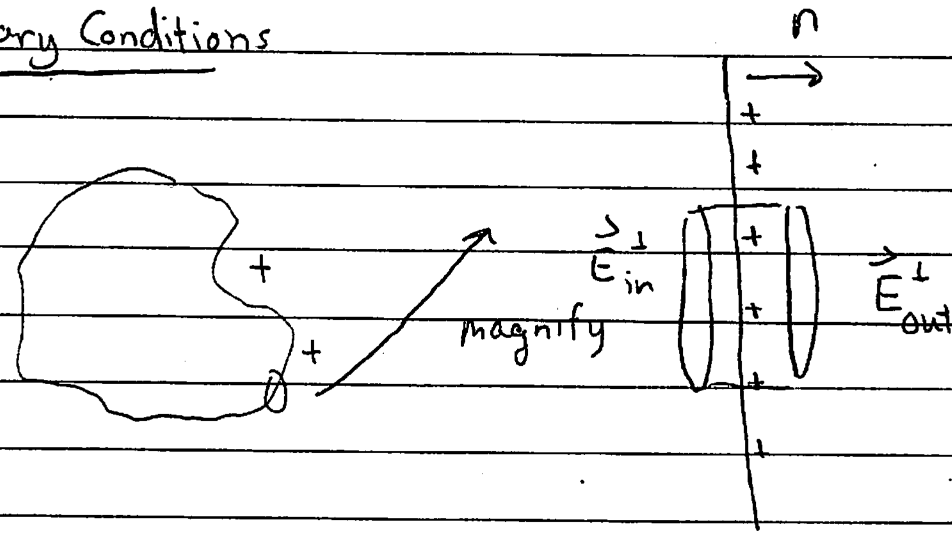
Substituting ☆ into $\nabla\cdot E = \rho(x)$ we find an equation for φ

$$-\nabla^2\varphi = \rho(x) \Leftarrow \text{Poisson equation}$$

In the absence of charge $\rho(x)$

$$-\nabla^2\varphi = 0 \Leftarrow \text{Laplace}$$

Boundary Conditions



Gauss Law:

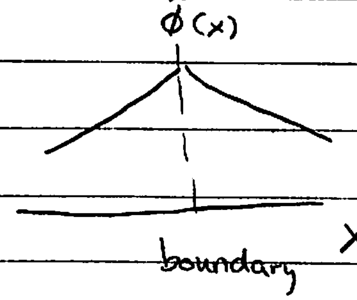
$$\int \vec{E} \cdot d\vec{a} = Q_{enc}$$

$$A\left(\vec{n} \cdot E_{out} - \vec{n} \cdot \vec{E}_{in}\right) = Q_{enc}$$

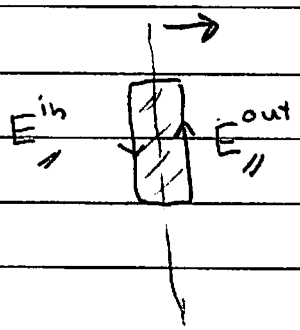
$$\boxed{E_{out}^{\perp} - E_{in}^{\perp} = \sigma}$$

$$\sigma \equiv \frac{Q}{A}$$

Picture



Similarly,



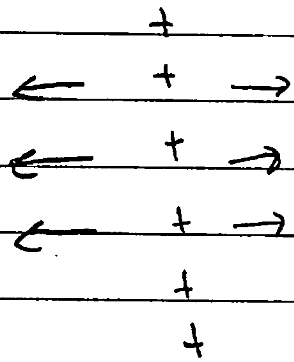
$$\int d\vec{a} \cdot \nabla \times \vec{E} = 0$$

$$\int d\vec{l} \cdot E = 0$$

$$\boxed{E_{\parallel}^{out} - E_{\parallel}^{in} = 0}$$

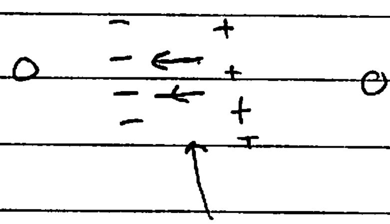
or $\quad n \times (E_{out} - E_{in}) = 0$

Charged Plane



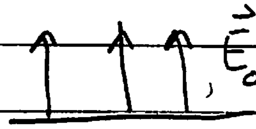
$$E = \frac{\sigma}{2}$$

For capacitor;



$E = \sigma$

Metal:

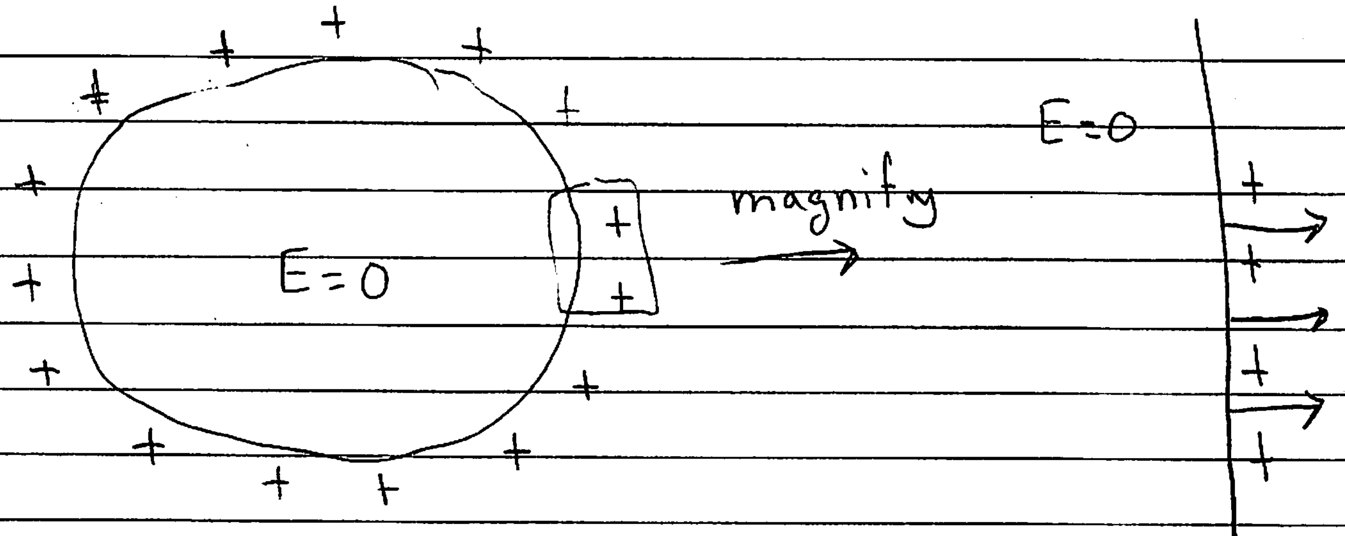


$\vec{E}_{out} = E_n \hat{n}$

$E_n = \sigma$

$E_{in} = 0$

Boundary Conditions + Force/Area on a metal plate



Ask about force per area

$$\frac{F}{A} = \sigma\big(\underbrace{E_{out} - E_{self}}_{\text{E-field not from self}}\big) \qquad E_{out} = \sigma$$

charge/area — the part of the electric field produced by σ

$$\frac{F}{A} = \sigma\big(\underset{E_{out}=\sigma}{\sigma} - \frac{\sigma}{2}\big)$$

from a wall of charge

E_{self} $\sigma/2$ + $\sigma/2$ = E_{self}

← + →

\+

\+

\+

$$\boxed{\frac{F}{A} = \frac{\sigma^2}{2}}$$

Energy and Forces

q_1 $\cdot$ $\quad$ q_4 $\cdot$

q_2 $\cdot$ $\quad$ q_3 $\cdot$

because we should sum over pairs

$$W_E = \frac{1}{2}\sum_i \sum_{j \neq i} \frac{q_i q_j}{4\pi |\vec{r}_i - \vec{r}_j|}$$

U_E is the energy required to assemble the charges.

When written in continuous form.

$$\boxed{\begin{aligned} W_E &= \frac{1}{2}\int_x \int_{x_0} \frac{\rho(\vec{x})\,\rho(\vec{x}_0)}{4\pi|\vec{x}-\vec{x}_0|} \\ &= \frac{1}{2}\int_x \rho(\vec{x})\,\varphi(x) \end{aligned}}$$

Now $\rho(\vec{x}) = -\nabla^2\varphi(x)$, so

$$W_E = \frac{1}{2}\int_x \left[-\partial_i\partial^i\varphi(x)\right]\varphi(x)$$

By parts. use $\varphi(x) \to 0$ as $x \to \infty$

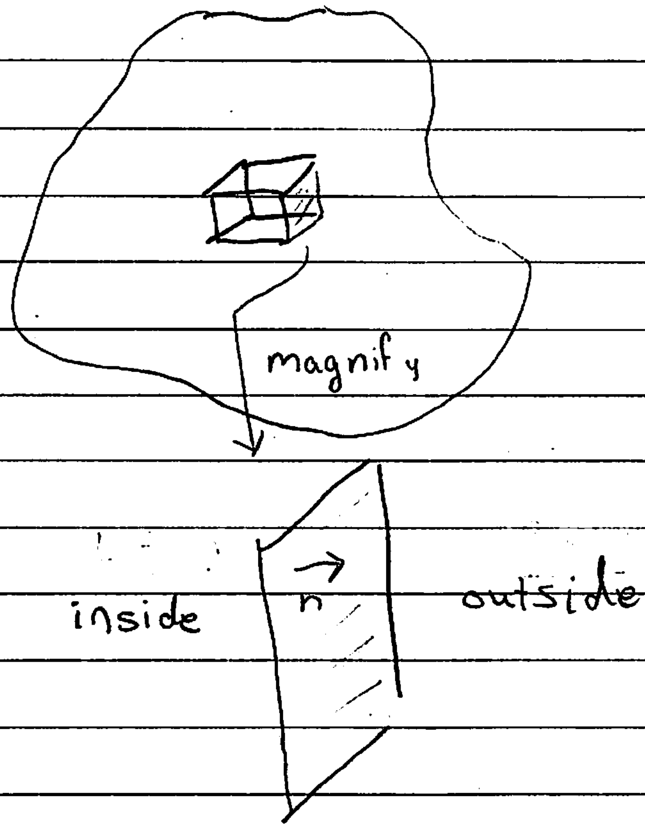
$$W_E = \frac{1}{2}\int_x \underbrace{\left[-\partial^i\varphi(x)\right]}_{E^i(x)} \cdot \underbrace{\left[-\partial_i\varphi(x)\right]}_{E_i(x)}$$

$$\boxed{W_E = \frac{1}{2}\int_x E^2(x)}$$

So the energy density is

$$\boxed{u_E = \frac{1}{2} E^2}$$

Force + Stress



T^{ij} = Force per area = stress tensor

= Force in the j-th direction
due to the area in the ith

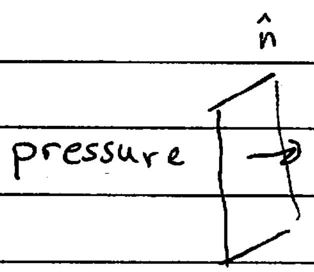
$n_i T^{ij}$ = Force in j-th direction on outside
by the inside

In general lots of forces mechanical (pressure)
in addition to Electrical!

Examples:

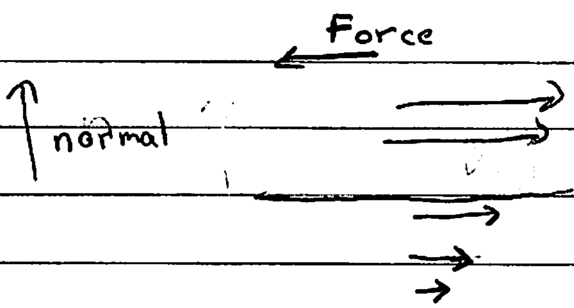
- Ideal Fluid at rest

$$T^{ij} = \overset{\text{pressure}}{p}\,\delta^{ij}$$



$$\frac{F^x}{A^x} = n_i T^{ix} = T^{xx} = p$$

- Viscous fluid:



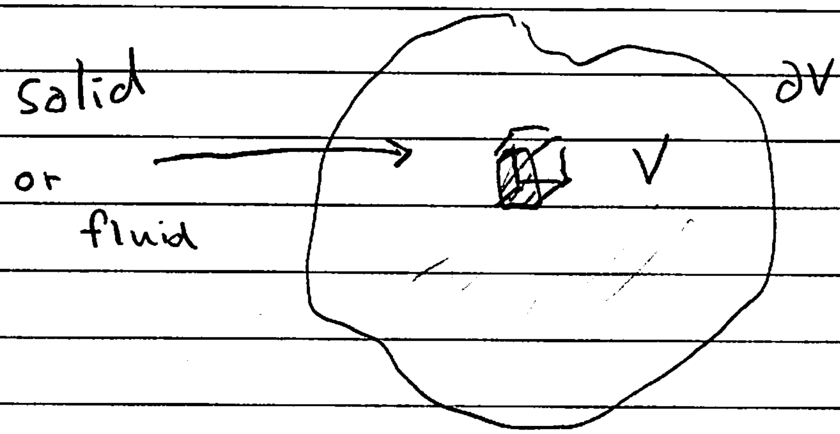
$$\frac{F^x}{A^y} = -\overset{\text{viscosity}}{\eta}\,\frac{\Delta v^x}{\Delta y} = T^{yx}$$

viscous force slows down upper stream

- Electricity, we will show that

$$T^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$$

Stress Tensor and momentum conservation laws



Take an element of solid and ask how the total momentum per volume $\equiv \vec{g}_{TOT}$ changes.

We would expect $\vec{g}_{TOT}$ to obey a conservation law:

$$\frac{\partial g^i_{TOT}}{\partial t} + \partial_i (T^{ij}) = 0 \Rightarrow \frac{\partial g^j}{\partial t} = \underbrace{-\partial_i T^{ij}}_{\text{force per volume in } j\text{-th direction}}$$

If $\vec{g}_{TOT}$ obeys an equation like this then:

$$\frac{\partial P^j_{TOT}}{\partial t} = \frac{\partial}{\partial t}\int_V d^3r\, g^j_{TOT} = -\int d^3r\, \partial_i T^{ij}$$

net force

$$= -\underbrace{\int_{\partial V} da\, n_i T^{ij}}_{\text{goes to zero if volume is to infinty}} \Rightarrow 0$$

Summary:

① $f^j = -\partial_i T^{ij}$ = force in j-th direction per volume
(difference in force per area)

② The total force on an object

$$\frac{dP^j_{TOT}}{dt} = -\int n_i T^{ij} \, da$$

Back to electrodynamics, the force is

$$f^j = \rho E^j \quad = \text{force per vol}$$

It should be the divergence of something:

$$f^j = \rho E^j \qquad \nabla \cdot E = \rho \iff \partial_i E^i = 0$$

$$f^j = (\partial_i E^i) E^j \qquad \nabla \times E = 0 \iff \partial_i E_j - \partial_j E_i = 0$$

In class problem

$$= \partial_i \underbrace{\left(E^i E^j - \frac{1}{2} \delta^{ij} E^2 \right)}_{-T^{ij}}$$

So

$$T^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$$

Solution to In Class Problems

(1) $f^j = \rho E^j$ $\quad$ $\nabla \cdot E = \rho \qquad \nabla \times E = 0$

$\partial_i E^i = \rho \qquad \partial_i E_j - \partial_j E_i = 0$

f^j should be the divergence of something:

$f^j = -\partial_i T^{ij} \longrightarrow$ what is T^{ij}?

Solution

$$f^j = \rho E^j$$

from $\partial_i E^j - \partial^j E_i = 0$

$$= (\partial_i E^i) E^j$$

$$= \partial_i (E^i E^j) - \underbrace{E^i \partial_i E^j}$$

$$= \partial_i (E^i E^j) - E^i \partial^j E_i$$

$\frac{1}{2} \partial^j (E^i E_i) = E^i \partial^j E_i$

$$= \partial_i (E^i E^j) - \frac{1}{2} \partial^j (E^i E_i)$$

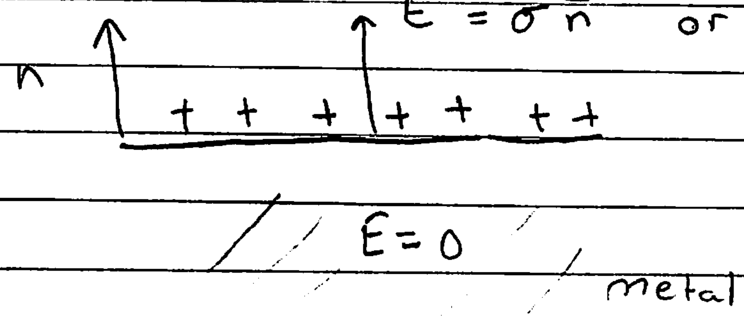
relabel $E^i E_i = E^2 = E^k E_k$

$$= \partial_i \left(E^i E^j - \frac{1}{2} \delta^{ij} E^2\right)$$

use $\partial_i \delta^{ij} = \partial^j$

$$= -\partial_i \underbrace{\left(-E^i E^j + \frac{1}{2} \delta^{ij} E^2\right)}_{T^{ij}}$$

For a metal



$\vec{E} = \sigma \vec{n}$ or $E^j = \sigma n^j$

$+ n_i T^{ij} = \frac{\text{force}}{\text{area}}$ on exterior by metal

$- n_i T^{ij} = \frac{\text{force}}{\text{area}}$ on metal by exterior

$$= -n_i \left(-E^i E^j + \frac{1}{2} E^2 \delta^{ij}\right)$$

$$= -n_i \left(-\sigma n^i \sigma n^j + \frac{1}{2} \sigma^2 \delta^{ij}\right)$$

$E^2 = \sigma \vec{n} \cdot \sigma \vec{n}$
$= \sigma^2$

$$- n_i T^{ij} = \sigma^2 n^j - \frac{1}{2} \sigma^2 n^j$$

$$\boxed{- n_i T^{ij} = \frac{\sigma^2}{2} n^j}$$

agrees with before for force/area