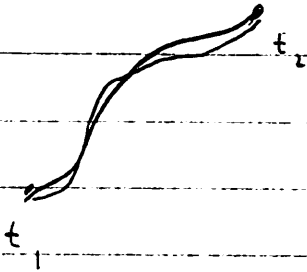


## A primer on actions

- Today's lecture will derive EOM from actions



$$I_0 = \int dt \frac{1}{2} m \dot{x}^2 \quad I_{int} = \int F_0 dx$$

The EOM are derived by varying the path  $x(t) \rightarrow x(t) + \delta x(t)$  and demanding that the action be stationary

$$\delta I = \delta I_0 + \delta I_{int}$$

$$= \int dt \delta x(t) [-m \ddot{x} + F]$$

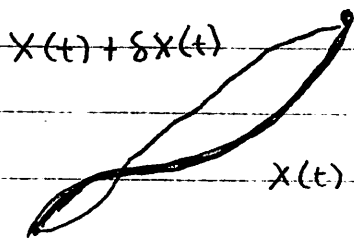
So EOM are

$$m \ddot{x} = F$$

$$\frac{dp}{dt} \rightarrow \underbrace{-\frac{\delta I_0}{\delta x}}_{\text{force}} \quad \underbrace{\frac{\delta I_{int}}{\delta x}}_{\text{force}}$$

Keep this in mind today

# Super Slow Discussion of Variations and Variational Derivs



Vary the path

$$x(t) \rightarrow x(t) + \delta x(t)$$

small

vanishes at end

Then

$$\frac{1}{2} m (\dot{x} + \delta \dot{x})^2 = \frac{1}{2} m \dot{x}^2 + m \dot{x} \delta \dot{x} + O(\delta x^2)$$

Now as  $x(t)$  is changed the integral changes,  $I_{\text{tot}} \rightarrow I_{\text{tot}} + \delta I$

$$I_{\text{tot}} \rightarrow I_{\text{tot}} + \delta I = \int \frac{1}{2} m (\dot{x} + \delta \dot{x})^2 + F(x + \delta x) dt$$

$$= I_{\text{tot}} + \int dt [m \dot{x} \delta \dot{x} + F \delta x] \quad \uparrow \text{by parts}$$

$$= I_{\text{tot}} + \int dt \left[ -\frac{d}{dt} (m \dot{x}) + F \right] \delta x(t)$$

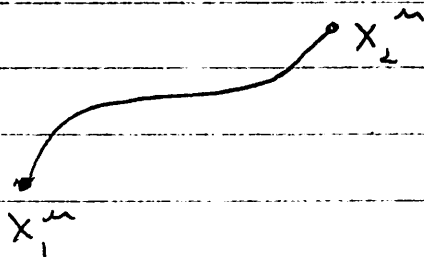
Now  $\delta x(t)$  is arbitrary. The only way to guarantee that  $\delta I_{\text{tot}} = 0$  is if the thing in square brackets vanishes.

• The thing in front of  $\delta x(t)$  is known as the variational derivative at time  $t$

$$\frac{\delta I_{\text{tot}}}{\delta x(t)} = -\frac{d}{dt} (m \dot{x}) + F$$

## The point particle Action pg. 1

For a point particle



$S = \int$  all possible forms allowed by Lorentz invariance  
and gauge invariance

Parametrize path by  $x^\mu(p)$  ← parameter

Then

$-\frac{dx^\mu}{dp} \frac{dx_\mu}{dp}$  is invariant

Action should also not depend on parametrization.

$$\frac{dx^\mu}{dp} \rightarrow \frac{dp'}{dp} \frac{dx^\mu}{dp'}$$

## Point Particle pg. 2

This restricts the action to:

$$I_0 = k \int dp \sqrt{\frac{-dx^\mu}{dp} \frac{dx_\mu}{dp}}$$

Some const, later take it to be  $-mc$

Often take  $p$  to be the proper time. Then

$$I_0 = -mc^2 \int d\tau \quad \text{where } c d\tau = \sqrt{-dx^\mu dx_\mu}$$

So we used

$$= dp \sqrt{\frac{-dx^\mu}{dp} \frac{dx_\mu}{dp}}$$

$$d\tau = \frac{dt}{\gamma} = dt \sqrt{1 - \dot{x}^2/c^2}$$

and thus, in the non-rel limit

$mc$  chosen so KE  
is  $\frac{1}{2}mv^2$

$$L_0 = -mc^2 (1 - (\dot{x}/c)^2)^{-1/2} \approx -mc^2 + \frac{1}{2}m\dot{x}^2$$

Now

$$\delta I_0 = -\delta \int dt mc^2 \sqrt{1 - \dot{x}^2/c^2}$$

$$= \int dt \frac{m \dot{x}}{\sqrt{1 - \dot{x}^2}} \partial_t \delta x$$

$$= - \int dt \delta x \left[ \frac{d}{dt} (\gamma m v) \right]$$

Point Particle pg. 3

So the EOM in absence of interactions

$$\frac{d}{dt} (\gamma m \vec{v}) = 0$$

## Covariant Formulation of Point Particle

$$I_0 = - \int mc \sqrt{-\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda}} d\lambda \quad c dt = \sqrt{-\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda}} d\lambda$$

Where we parametrize the path @  $\lambda$ . So the variation  $x^\mu \rightarrow x^\mu + \delta x^\mu$  gives

$$\delta I_0 = \int \frac{-mc}{\left(-\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda}\right)^{+1/2}} \cdot \frac{-dx^\mu}{d\lambda} \frac{d\delta x_\mu}{d\lambda}$$

↑ integrate by parts

Then

$$\star) \delta I_0 = - \int d\lambda \left[ \frac{d}{d\lambda} \frac{mc}{\sqrt{-\dot{x} \cdot \dot{x}}} \frac{dx^\mu}{d\lambda} \right] \delta x_\mu$$

In terms of proper time:

$$d\lambda \frac{d}{d\lambda} = dt \frac{d}{dt} \quad \text{with} \quad \frac{d}{dt} = \frac{c}{\sqrt{-\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda}}} \frac{d}{d\lambda}$$

So Eq  $\star$  reads:

$$\delta I_0 = - \int dt \left[ \frac{d}{dt} \left( m \frac{dx^\mu}{dt} \right) \right] \delta x_\mu$$

So in absence of interactions, EOM are  $m \frac{d^2 x}{dt^2} = 0$

## The interaction Lagrangian

$$I_{\text{int}} = \frac{e}{c} \int d\lambda \frac{dx^\mu}{d\lambda} A_\mu \leftarrow \text{invariant under reparametrization}$$

↑ only Lorentz invariant linear in fields  $F_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$

or

$$= \frac{e}{c} \int d\tau \frac{dx^\mu}{d\tau} A_\mu$$

or

$$I_{\text{int}} = \int dt \left[ -e\varphi + \frac{d\vec{x}}{dt} \cdot \frac{\vec{A}}{c} \right]$$

$$d\tau = \frac{dt}{\gamma} \quad \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \vec{v})$$

↑ most usual form in non-rel context

So,  $A^\mu = A^\mu(x(\lambda))$

$$\delta I_{\text{int}} = \frac{e}{c} \int d\lambda \frac{d}{d\lambda} \delta x^\mu A_\mu + \frac{dx^\mu}{d\lambda} \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu$$

$$= \frac{e}{c} \int d\lambda \delta x^\mu \left( -\frac{\partial A_\mu}{\partial \lambda} \right) + \frac{dx^\mu}{d\lambda} \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu$$

Using

$$\frac{\partial A_\mu}{\partial \lambda} = \frac{\partial A_\mu}{\partial x^\rho} \frac{dx^\rho}{d\lambda}$$

Re labelling Contracted indices

$$\delta I_{int} = \int d\lambda \begin{bmatrix} \frac{\partial A_\alpha}{\partial x^\beta} & -\frac{\partial A_\beta}{\partial x^\alpha} \end{bmatrix} \frac{dx^\alpha}{d\lambda} \delta x^\beta$$

$$\begin{aligned} \delta I_{int} &= \int d\lambda F_{\beta\alpha} \frac{dx^\alpha}{d\lambda} \delta x^\beta \\ &= \int d\tau F_{\beta\alpha} \underbrace{\frac{dx^\alpha}{d\tau}}_{=u^\alpha} \delta x^\beta \end{aligned} \quad \left. \begin{array}{l} \uparrow \\ \downarrow \end{array} \right\} dp \frac{d}{dp} = d\tau \frac{d}{d\tau}$$

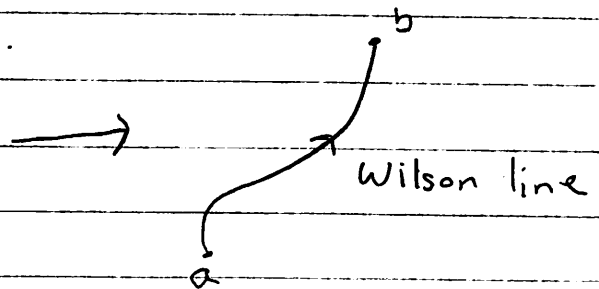
So then

$$\delta I_{tot} = \delta I_0 + \delta I_{int}$$

$$= \int d\tau \left[ -m \frac{d^2 x}{d\tau^2} + F_{\beta\alpha} u^\alpha \right] \delta x^\beta$$

Summary @ view to quantum mechanics

$$e^{iS[x]} = e^{i\frac{e}{\hbar} \int_a^b dx_\mu A^\mu}$$





## Hamiltonian Formulation

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - e\varphi + \frac{e\mathbf{v} \cdot \mathbf{A}}{c}$$

To construct the hamiltonian we first construct the canonical momentum

$$\vec{P}_{\text{can}} = \frac{\delta S}{\delta \dot{\mathbf{x}}} = \frac{\partial L}{\partial \mathbf{v}} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} + e\frac{\vec{A}}{c} = \vec{p}_{\text{kin}} + e\frac{\vec{A}}{c} = \vec{P}_{\text{can}}$$

Here  $\vec{p}_{\text{kin}} = m\gamma\vec{v}$  is the kinetic momenta. Sometimes I will stop writing "kin". Then

$$H = \vec{P}_{\text{can}} \cdot \mathbf{v} - L$$

$$H = \frac{mc^2}{\sqrt{1 - \beta^2}} + e\varphi$$

do it!

Now for the Hamiltonian framework one should express  $\beta$  in terms of  $\vec{P}_{\text{can}}$ . Using

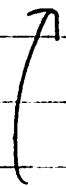
$$\left(\vec{P}_{\text{can}} - \frac{e\mathbf{A}}{c}\right)^2 = -(mc)^2 + \frac{(mc)^2}{1 - \beta^2}$$

We have

$$H = c \sqrt{\left( P_{\text{can}} - \frac{eA}{c} \right)^2 + (mc)^2} + e\psi$$

In the non-rel limit  $mc \gg P_{\text{can}} - \frac{eA}{c}$  and find

$$H \approx mc^2 + \frac{\left( P_{\text{can}} - \frac{eA}{c} \right)^2}{2m} + e\psi$$



This is much easier to derive from the non-relativistic Lagrangian:

$$L = \frac{1}{2} m v^2 - e\psi + \frac{e\vec{V} \cdot \vec{A}}{c}$$

$$P_{\text{can}} = m\vec{v} + \frac{e\vec{A}}{c}$$

$$H = p\dot{q} - L = \frac{\left( P_{\text{can}} - \frac{e\vec{A}}{c} \right)^2}{2m} + e\psi = H$$