A Lagrangian for the Fields

\[ I[A] = \int \text{all possible lorentz invariants.} \]

consistent with symmetries and no more than quadratic

Given field strength \( F^{\mu\nu} \) there are two possible Lorentz invariant forms

1. \( F_{\mu\nu} F^{\mu\nu} = 2 (B^2 - E^2) \)
2. \( F_{\mu\nu} \tilde{F}^{\mu\nu} = -4 E \cdot B \)

So the general form is \( I[A] = \int d^4 x \ a_1 F_{\mu\nu} F^{\mu\nu} + a_2 \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \)

This is parity odd.

Choose this coefficient to be \( -\frac{1}{4} \)

The factor \( \frac{1}{4} \) is conventional.

The (-1) in \( -\frac{1}{4} \) is chosen so that we have,

\[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2) \]

like PE

\[ \sim (\nabla \times A)^2 \]

\[ \sim (\partial_t \vec{A})^2 \] like kinetic energy
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Then

\[ I_o = \int d^4x \ - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

write \( A_\mu \rightarrow A_\mu + \delta A_\mu \)

and expand see next pages

\[ \delta I_o = \int d^4x \ - \frac{1}{2} F_{\mu\nu} \left( \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \right) \]

by parts

\[ = \int d^4x \ + \frac{1}{2} \left( \partial_\mu F^{\mu\nu} \delta A_\nu - \partial_\nu F^{\mu\nu} \delta A_\mu \right) \]

Relabel and

\[ = \int d^4x \ \delta A_\alpha \left[ \partial_\alpha F^{\alpha\beta} \right] \]

use antisymmetric of \( F^{\mu\nu} \)

In general the field is coupled to currents

\[ I_{\text{Int}} = \int d^4x \ J^\mu A_\mu \]

For example for the particle lagrangian

\[ I_{\text{Int}} = \int dt \ e \frac{dx^\mu}{dt} \ A_\mu \]
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In general, define the current as

$$\delta I_{\text{int}} = \int d^4 x \ J^\mu \ \delta A_\mu$$

or this is written,

$$\frac{\delta I_{\text{int}}}{\delta A_\mu(x)} = J^\mu(x)$$

Then

$$\delta I_o + \delta I_{\text{int}} = \int d^4 x \ \delta A_\beta \left[ \partial_\alpha F^{\alpha\beta} + J^\beta \right]$$

leading to the field eqs

$$\partial_\alpha F^{\alpha\beta} = J^\beta$$

$$\frac{\delta I_o}{\delta A_\beta} = \frac{\delta I_{\text{int}}}{\delta A_\beta}$$

analogous to
ma
analogous to force.
Slow motion variation of $F^2$

\[ \delta F^2 = \delta (F^{\mu\nu} F_{\mu\nu}) = \delta F^{\mu\nu} F_{\mu\nu} + F^{\mu\nu} \delta F_{\mu\nu} \]

\[ = \delta F_{\mu\nu} F^{\mu\nu} + F^{\mu\nu} \delta F_{\mu\nu} \]

\[ = 2 F^{\mu\nu} \delta F_{\mu\nu} \]

Now

\[ \delta F_{\mu\nu} = (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) \]
Gauge invariance & Current Conservation

Consider the interaction between the currents and the Maxwell field

\[ J_{\text{int}}[A] \]

Let's assume that this interaction is gauge invariant.

\[ \delta J_{\text{int}} = \frac{1}{c} \int d^4x \ J^m \delta A_m \]

Now, if I make a gauge transformation, this does not change the value of \( J_{\text{int}}[A] \) or \( \delta J_{\text{int}}[A] \).

Since the interaction is gauge invariant,

\[ \delta A^\mu \rightarrow \delta A^\mu + \frac{\partial \delta \Lambda}{\partial x^\mu} \]

Then,

\[ \delta J_{\text{int}} = \frac{1}{c} \int d^4x \ J^m \delta A_m + \frac{1}{c} \int d^4x \ J^m \frac{\partial \delta \Lambda}{\partial x^m} \]

So,

\[ 0 = \frac{1}{c} \int d^4x \ J^m \frac{\partial \delta \Lambda}{\partial x^m} \]
Integrating by parts

\[ 0 = - \int d^4x \frac{\partial J^m}{\partial x^n} \delta \lambda = \frac{\partial J^m}{\partial x^n} = 0 \]
In class problems

* Work in the confines of electrostatics. Show that the action takes the form

\[ I = \int d^4x \frac{1}{2} (-\nabla \phi)^2 - \rho \phi \]

And that the variation of the action with respect to \( \phi \) gives the Poisson Eq.:

\[ -\nabla^2 \phi = \rho \]

* The interaction lagrangian of a point particle is, where \( x_0(t) \) is the trajectory of the particle

\[ I_{int} = \frac{\varepsilon}{c} \int d\tau \frac{dx_0^\mu}{d\tau} A_\mu(x_0(t)) \]

Show that the current is

\[ J^\mu = (\varepsilon, \varepsilon) = (ecS^3(x-x_0(t)), eV \delta^3(x-x_0(t))) \]

Hint, start by writing

\[ I_{int} = \int d^4x \int d\tau \frac{dx_0^\mu}{d\tau} \partial_\mu(x_0) \delta^4(x - x_0(t)) \]

and then vary \( A_\mu(x) \).
Solution (1)

\[- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2) = \frac{1}{2} E^2 = \frac{1}{2} (-\nabla \phi)^2\]

Similarly

\[\int d^4x \frac{J^m}{c} A^m = \int d^4x \left[ \frac{(c \rho)}{c} (-\phi) + \frac{\hat{A}}{c} \cdot \hat{A} \right]\]

we used \(J^m = (c \rho, \hat{j})\), \(A^m = (\varphi, \hat{A})\), \(A^m = (-\varphi, \hat{A})\)

So

\[\int d^4x \frac{J^m}{c} A^m = -\int d^4x \rho \phi\]

Then

\[I_{\text{tot}} = \int d^4x \frac{1}{2} (-\nabla \phi)^2 - \rho \phi\]

\[\delta I = \int d^4x \left[ \nabla \phi \nabla \delta \phi - \rho \delta \phi \right]\]

\[= \int d^4x \delta \phi \left[ -\nabla^2 \phi - \rho \right]\]

Or

\[-\nabla^2 \phi = \rho\]
Solution (2)

\[ \delta I_{int} = \frac{e}{c} \int d^4x \left[ \int d\tau \frac{d\mathbf{x}_\alpha}{d\tau} \delta^4(x - \mathbf{x}_\alpha(\tau)) \right] \delta \mathbf{A}_\mu(x) \]

So

\[ \mathbf{J}^\mu = \frac{e}{c} \int d\tau \frac{d\mathbf{x}_\alpha}{d\tau} \delta^4(x - \mathbf{x}_\alpha(\tau)) \]

Then we integrate over \( \mathcal{T} \), \( d\mathcal{T} = \frac{dt}{\gamma} \), with
\[ \frac{d\mathbf{x}_\alpha}{d\tau} = (\gamma \mathbf{v}, \gamma v^t) \]

\[ \mathbf{J}^\mu = e \int dt (\gamma \mathbf{v}, \gamma v^t) \delta^4(x - \mathbf{x}_\alpha(t)) \]

\[ \mathbf{J}^\mu = (e\gamma \delta^3(x - \mathbf{x}_\alpha(t)), e\gamma v \delta^3(x - \mathbf{x}_\alpha(t))) \]