Last Time

* Finished by discussing the stress tensor:

\[
\Theta^{\mu\nu}_{tot} = \begin{pmatrix}
U_{tot} & \tilde{\tau}_{tot} / c \\
\tilde{\tau}_{tot} / c & T^i i
\end{pmatrix}
\]

\[\exists \Theta^{\mu\nu}_{tot} = 0\]

\[E\text{-conserv} \quad \text{0-component}\]

\[\Theta^{00}_{tot} = \text{energy density} = U_{tot} \quad \exists \Theta^{0\mu}_{tot} = 0\]

\[\Theta^{0i}_{tot} = \text{energy flux} = \tilde{S}/c = \tilde{g}c\]

\[M\text{om-conserv} \quad \text{i-th component}\]

\[\Theta^{i0}_{tot} = \text{momentum density} = \tilde{g}c = \tilde{S}/c \quad \exists \Theta^{i\mu}_{tot} = 0\]

\[\Theta^{ij}_{tot} = \text{stress force/area} = T^{ij}\]

If I have a mechanical system (like a fluid), with currents then the E+M fields will push and pull the system:

\[\exists \Theta^{\mu\nu}_{mech} = F^\nu_p \frac{J^\nu}{c} \quad \exists \Theta^{0\mu}_{mech} = E \cdot \frac{\tilde{g}}{c}\]

\[\exists \Theta^{i\mu}_{mech} = \rho E^i + \frac{(J \times B)}{c}\]

And thus mechanical energy and momentum won't be conserved.
Last Time

The electromagnetic force must be the divergence of something:

\[ F^\mu \frac{\mathcal{J}^\rho}{c} = - \partial_\mu \Theta^{\mu \nu}_{\text{em}} \]

Homework: show using \(-\partial_\mu F^{\mu \nu} = J^\nu / c\) that

\[ \Theta^{\mu \nu}_{\text{em}} = F^{\nu \lambda} F^\lambda_{\mu} + g^{\mu \nu} \left( -\frac{1}{4} F^2 \right) \] (see below)

Then

\[ \partial_\mu \Theta^{\mu \nu}_{\text{mech}} = - \partial_\mu \Theta^{\mu \nu}_{\text{em}} \]

or

\[ \partial_\mu (\Theta^{\mu \nu}_{\text{mech}} + \Theta^{\mu \nu}_{\text{em}}) = 0 \]

and thus the combined mechanical + electromagnetic energy and momentum will be conserved.

\[ \Theta^{\mu \nu}_{\text{em}} = \begin{pmatrix} \frac{1}{2} (E^2 + B^2) \\ \frac{E \times B}{2} \\ \frac{-E \cdot E + \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} E^\rho E^\sigma}{2} \\ \frac{-B \cdot B + \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} B^\rho B^\sigma}{2} \end{pmatrix} \]

\[ \begin{pmatrix} \mathcal{U}_{\text{em}} \\ \mathcal{E}_{\text{em}} \\ \mathcal{S}_{\text{em}} c \end{pmatrix} = \begin{pmatrix} \mathcal{U}_{\text{mech}} \\ \mathcal{S}_{\text{mech}} c \end{pmatrix} \]
Radiation From Relativistic Charges

\[ \begin{align*}
\mathbf{r}(t, \mathbf{r}) & \quad \text{Observation point} \\
(t, \mathbf{r}) & \quad \text{Point} \\
(t_0, \mathbf{r}_0(t_0)) & \quad \text{Source point}
\end{align*} \]

Using the Green func

\[ - \Delta \Phi = \rho \]
\[ - \nabla^2 \mathbf{A} = \frac{j}{c} \]
\[ G(t, r | t_0, r_0) = \Theta(t - t_0) \delta(t - t_0 - \frac{r - r_0}{c}) \frac{1}{4\pi |r - r_0|} \]

So

\[ \Phi(t, \mathbf{r}) = \int d\mathbf{r}_0 \int d\mathbf{r}_0 \, \delta(t - t_0 - |\mathbf{r} - \mathbf{r}_0|) \frac{\rho(t, \mathbf{r}_0)}{c} \frac{1}{4\pi |r - r_0|} \]
\[ \mathbf{A}(t, \mathbf{r}) = \int d\mathbf{r}_0 \int d\mathbf{r}_0 \, \delta(t - t_0 - |\mathbf{r} - \mathbf{r}_0|) \frac{j}{c} \frac{1}{4\pi |r - r_0|} \]

For point charge

\[ \rho(t, \mathbf{r}_0) = e \delta^3(\mathbf{r}_0 - \mathbf{r}_0(t_0)) \]
\[ \mathbf{j}(t, \mathbf{r}_0) = e \mathbf{v}(t_0) \delta^3(\mathbf{r}_0 - \mathbf{r}_0(t_0)) \]
Doing the \( d^3r \) integral

\[
\mathcal{E}(r, r') = \int dt_0 \frac{\delta \left( t - t_0 - \frac{\left| r - r_*(t_0) \right|}{c} \right)}{\pi \pi \pi R} \frac{e^\frac{\left| r - r_*(t_0) \right|}{c}}{\pi \pi \pi R}
\]

\[
\mathcal{A}(r, r') = \int dt_0 \frac{\delta \left( t - t_0 - \frac{\left| r - r_*(t_0) \right|}{c} \right)}{\pi \pi \pi R} \frac{e^\frac{\left| r - r_*(t_0) \right|}{c}}{\pi \pi \pi R}
\]

Now we do the time integral, for each value of \( \bar{r}, \bar{r}' \) only one time moment \( t_0 = T \) will contribute.

\[
T = t - \frac{\left| \bar{r} - r_*(T) \right|}{c} = \text{retarded time or source time}
\]

Using \( \delta (f(t_0)) = \delta \left( t_0 - \frac{T}{c} \right) \) with \( f(t_0) = t - t_0 - \frac{\left| \bar{r} - r_*(t_0) \right|}{c} \)

we have

\[
\frac{df}{dt_0} = -1 - \frac{1}{c} \frac{d}{dt_0} \left( \frac{\left( \bar{r} - r_*(t_0) \right)^2}{c^2} \right)^{\frac{1}{2}} = -1 + \vec{n} \cdot \vec{V}(t_0)
\]

with \( \vec{n} = \frac{\bar{r} - r_*(t_0)}{\left| \bar{r} - r_*(t_0) \right|} \)

And so

\[
\frac{1}{\left| f'(T) \right|} = \frac{1}{\left( 1 - \frac{n \cdot V(T)}{c} \right)}
\]
So we are led to

\[ \Phi(t, \hat{r}) = \frac{e}{4\pi R \left( 1 - \hat{n} \cdot \hat{v}(T) \right)} \]

\[ \hat{A}(t, \hat{r}) = \frac{e \hat{v}(T)/c}{4\pi R \left( 1 - \hat{n} \cdot \hat{v}(T) \right)} \]

These are known as the Lienard - Wiechert Potentials

Let us specialize to the far field

\[ \frac{1}{R} \sim \frac{1}{\Gamma} \]

\[ T = t - \frac{1}{c} |\hat{r} - \hat{r}_k(T)| \sim t - \frac{\hat{n} \cdot (\hat{r} - \hat{r}_k(T))}{c} = T \]

\[ \hat{n} \sim \frac{\hat{r}}{|\hat{r}|} \]

retarded time in far field

this is an implicit function of \( t, \tau \)
Problem

* Show that \( \frac{\partial T}{\partial t} = \frac{1}{c^2} \left( 1 - \frac{\mathbf{n} \cdot \mathbf{v} c}{c^2} \right) \quad \text{and} \quad \frac{\partial T}{\partial r_i} = \frac{1}{c} \left( 1 - \frac{\mathbf{n} \cdot \mathbf{v} c}{c^2} \right) \)

Interpret \( \frac{\partial T}{\partial t} \) physically by drawing a picture.

Solution - use implicit differentiation

1. \( T = \frac{c}{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_*(t))} \)

\[
\frac{\partial T}{\partial t} = \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{1}{c} \left( 1 - \mathbf{n} \cdot \mathbf{v} c \right) \right) \Rightarrow \frac{\partial T}{\partial t} = \frac{1}{c} \left( 1 - \frac{\mathbf{n} \cdot \mathbf{v} c}{c^2} \right)
\]

velocity

2. \( \frac{\partial T}{\partial r_i} = -\frac{\mathbf{n}_i c}{\mathbf{r} \cdot \mathbf{v}} + \frac{\mathbf{n} \cdot \mathbf{v} c}{c^2} \frac{\partial}{\partial r_i} \left( \frac{1}{c} \left( 1 - \frac{\mathbf{n} \cdot \mathbf{v} c}{c^2} \right) \right) \)

So then we have an interpretation of \( \frac{1}{c} \left( 1 - \frac{\mathbf{n} \cdot \mathbf{v} c}{c^2} \right) \).

First note that this factor can be very large if the observation direction is parallel to \( \mathbf{v} \) and \( \mathbf{v} \approx c \).
Physical interpretation of $\partial T/\partial t = \frac{1}{(1 - \vec{v} \cdot \vec{u}/c)}$ observer.

Particle first flying, wave emitted.

Second wave emitted position of first wave.

So the observer measures the time difference between the signals to be:

$$\Delta t = \frac{(c - v) \Delta T}{c}$$

$$\Delta t = \frac{(1 - v) \Delta T}{c}$$

$$\frac{1}{(1 - v)} \Delta t = \Delta T$$

So

$$\Delta \bar{T} = \text{formation time of radiation} = \frac{1}{(1 - \hat{n} \cdot \hat{v}/c)}$$

$$\frac{\Delta t}{\text{observation time of radiation}}$$
Fields of Lienard-Wiechert

Now we can compute the Electric Field

\[ E_{\text{rad}} = \hat{n} \times \hat{n} \times \frac{1}{c} \frac{2A_{\text{rad}}}{\partial t} \]

So first we relate \( \frac{\partial A}{\partial t} \) and \( \frac{\partial A}{\partial T} \):

\[ \frac{1}{c} \frac{\partial A_{\text{rad}}}{\partial t} = \frac{1}{c} \frac{\partial A_{\text{rad}}}{\partial T} \frac{\partial T}{\partial t} - \frac{1}{c} \frac{\partial A_{\text{rad}}}{\partial T} \frac{1}{1 - \hat{n} \cdot \hat{V}/c} \]

Using:

\[ A_{\text{rad}} = \frac{e}{4\pi r} \frac{\hat{V}(T)/c}{(1 - \hat{n} \cdot \hat{V}/c)} \]

\[ \frac{1}{c} \frac{\partial A_{\text{rad}}}{\partial T} = \frac{e}{4\pi rc^2} \frac{\hat{a}(T)}{(1 - \hat{n} \cdot \hat{V}/c)} - \frac{e}{4\pi rc^2} \frac{\hat{b} \cdot (\hat{n} \cdot \hat{a})}{(1 - \hat{n} \cdot \hat{V}/c)^2} \]

\[ = \frac{e}{4\pi rc^2} \frac{1}{(1 - \hat{n} \cdot \hat{V}/c)^2} \left[ \hat{a} + \hat{n} \times \hat{b} \times \hat{a} \right] \]

Then \( \hat{n} \times \hat{n} \times \frac{1}{c} \frac{\partial A_{\text{rad}}}{\partial t} \) is already transverse

\[ E_{\text{rad}} = \frac{e}{4\pi rc^2} \frac{1}{(1 - \hat{n} \cdot \hat{V}/c)^3} \left[ \hat{n} \times (\hat{n} - \hat{b}) \times \hat{a} \right] \]
\[ B_{\text{rad}} = n \times E_{\text{rad}} \]

It is understood that \( V(T) \) and \( a(T) \) are to be evaluated at the retarded time

\[ T = t - \frac{\hat{n} \cdot (\hat{r} - r_*(T))}{c} \]

The Power
Last Time

Talked About the Lienard-Wiechert potential

\[ (T, r^* (T)) \quad \hat{n} \cdot (t, r) \quad T = t - \frac{|T - r^*(t)|}{c} \]

For a relativistic particle solved

\[ \Box \psi = \rho \]
\[ \Box \hat{A} = \frac{J}{c} \]

To find:

\[ \psi (y) = \frac{e^{-i \frac{y - r^*(T)}{c}}}{4\pi |y - r^*(T)| (1 - \hat{n} \cdot \beta (T))} \]

\[ \hat{A} (t, r) = \frac{e^{-i \frac{y - r^*(T)}{c}}}{4\pi |y - r^*(T)| (1 - \hat{n} \cdot \beta (T))} \frac{V (T)}{c} \]

Here \( T = t - \frac{|T - r^*(t)|}{c} \) = retarded time, and

\[ \hat{n} = \frac{\hat{T} - \hat{r}^*(T)}{|\hat{T} - \hat{r}^*(T)|}, \quad \text{and} \quad \hat{\beta} (t) \equiv \frac{V (t)}{c} \equiv \frac{1}{c} \frac{d \hat{r}^*(t)}{dT} \]
Last Time pg. 2
(Aside:)

One can write it covariantly

\[ \Delta X^m = (c(t-T), \vec{r} - \vec{r}_*(r)) = \text{observation - emission point} \]

Then

\[ u^m = (\delta c, \delta \vec{v}) \quad \Delta X^m \Delta x_\mu = 0 \]

So

\[ A^m = - e \frac{u^m}{c} \quad \text{(end aside)} \]

\[ \frac{1}{\gamma \Pi \tau} \cdot u \cdot \Delta x \]

Then in the far field

\[ T = t - \frac{\vec{n} \cdot (r - r_*(r))}{c} \]

\[ y = \frac{e^{-1}}{\gamma \Pi \tau (1 - \vec{n} \cdot \vec{v}(\tau)/c)} \]

\[ \dot{A} = e \frac{\vec{v}(\tau)/c}{\gamma \Pi \tau (1 - \vec{n} \cdot \vec{v}(\tau)/c)} \]

Computing \( \vec{E} \)

\[ \vec{E} = - e \frac{\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{a}}{\gamma^3 \Pi \tau c^2 (1 - \vec{n} \cdot \vec{\beta}(\tau))^3} \]

\[ \vec{B} = \vec{n} \times \vec{E} \]
We also discussed the origin of the "collinear factor":

\[ \Delta T(t,r) = \frac{1}{2t} \frac{1}{1 - \hat{n} \cdot \hat{v}(T)/c} \]

Then we said, using kinematics:

\[ \Delta T = \frac{\Delta t}{(1 - \hat{n} \cdot \hat{v}(T)/c)} \]

That is, if the radiation pattern is formed over time \( \Delta T \), then it will be observed to have time scale \( \Delta T \). The formation time is related to \( \Delta t \) by:
Radiated Power

\[ \frac{dP(r)}{d\Omega \, dt \, ds} = \frac{e^2}{\lambda} \hat{S} \cdot \hat{n} \]

This is what you want to know if you want to know if the detector will burn up.

One often wants to know how much energy was radiated away as the particle moved from \( A \) (labelled by \( (T_A, \gamma(T_A)) \)) and \( B \) (labelled by \( (T_B, \gamma(T_B)) \)). Then you want to know

\[ \frac{dP(T)}{d\Omega} = \frac{dW}{dT \, ds \, ds} = \frac{dW}{dt \, d\Omega \, d\Omega} \]

Using

\[ \frac{dT}{dt} = \frac{1}{(1 - n \cdot \beta(t))} \]

\[ S = c \, E^2 \frac{1}{\lambda} \]

We have

\[ \frac{dP(T)}{d\Omega} = \frac{|r \cdot E|^2 (1 - n \cdot \gamma(t)/c)}{16 \pi^2 c^3} \left( \frac{1}{(1 - n \cdot \beta(t))^5} \right) \]
Radiated Power $\hat{a}$ parallel to $\hat{b}$

Then let's take the simplest case. A particle moving relativistically but decelerating along the motion:

\[
\hat{\alpha} \cdot \hat{\beta} = \beta \cos \theta
\]

\[
|\hat{\alpha} \times \hat{\alpha} \times \hat{\alpha}| = \alpha = \alpha \sin \theta
\]

So

\[
dP(\Omega) = \frac{e^2 \alpha^2 \sin^2 \theta}{d\Omega \cdot \frac{16\pi^2 \epsilon^3}{(1 - \beta \cos \theta)^5}}
\]

Comments:

1. For non-relativistic motion, get the Larmor result $\theta \ll 1$

\[
dP(\Omega) = \frac{e^2 \alpha^2 \sin^2 \theta}{d\Omega \cdot \frac{16\pi^2 \epsilon^3}{}}
\]

2. For $\beta \rightarrow 1$ and $\theta \rightarrow 0$, $(1 - \beta \cos \theta)$ gets large and the radiation is peaked in the direction of motion. For $\delta = 2$ this is plot $\theta \sim \frac{\pi}{2}\theta$

← Polar Plot of $\frac{dP}{d\Omega}$
Radiated Power \( \alpha \gamma^{2} \) pg. 2

Take the limit \( \beta \to 1 \), \( \theta \) small, \( \gamma^{2} = \frac{1}{(1-\beta^{2})} = \frac{1}{2(1-\beta)} \)

\[
\frac{1}{(1-\beta \cos \theta)} \sim \frac{1}{(1-\beta) + \theta^{2}} \frac{2 \gamma^{2}}{2 \gamma^{2} + \theta^{2}}
\]

So then with \( \sin \theta = \theta \)

\[
\frac{dP}{d\Omega} = \frac{2e^{2}}{\gamma^{2}} \frac{a^{2}}{\pi^{2} c^{3}} \frac{\gamma^{8} (\gamma \theta)^{2}}{(1 + (\gamma \theta)^{2})^{5}}
\]

So the picture is take \( \gamma \) large \( \sim 100 \). Then the radiation is peaked in the forward direction \( \Theta \sim \frac{1}{100} \).

But only transverse currents radiate. So in the direction of motion of the particle, \( \frac{1}{2} \gamma \theta \) there is no radiation. This is known as the deadcone, and is characteristic of heavy quark jets.

\[\text{dead cone}\]

\[\text{collinear radiation}\]

\[\text{dead cone}\]
Total Radiated Power \(a \parallel \text{to } \beta\)

We can also compute the total power:

\[ dP = 2\pi \sin \theta \, d\theta \approx 2\pi \, d\theta \quad (\theta < 1) \]

So

\[
P = \int d\Omega \, dp = \frac{2e^2}{\pi} \frac{a^2}{c^3} \frac{1}{\pi^2} \int_0^\pi r \, d\theta \frac{r \theta}{(1 + (r \theta)^2)^2}
\]

Let \( x = r \theta \)

\[
P = \frac{4e^2}{\pi} \frac{a^2}{c^3} \int_0^{\pi/2} x \, dx \frac{x^2}{(1 + x^2)^2}
\]

Now you can extend the upper limit \(\pi \to \infty\) \((\gamma \text{ large})\) and find

\[
P \approx \frac{e^2}{\pi} \left( \frac{2}{3} \right) \frac{a^2}{c^3} \gamma^6
\]

We will see that this is a special case of a relativistic generalization of the Larmor formula. Note that I put \(a \parallel\) because I have assumed that the acceleration is parallel to the velocity. In general,
the acceleration has a component parallel to the velocity \( a_\parallel \) and perpendicular to the velocity \( a_\perp \).

The full generalization of Larmor is (see below)

\[
P(T) = \frac{e^2}{4\pi^2} \frac{\gamma^6}{3c^3} \left[ \frac{a^2_\parallel + a^2_\perp}{\gamma^2} \right]
\]

Lienard–Wiechert 1898, predating relativity by seven years!

Proof of Lienard–Wiechert · (Brute force)

(Skip if pressed for time!)

\[
P(T) = \int d\Omega \frac{e^2}{16\pi^2c^3} \frac{1}{(1 - \kappa \cdot \beta)^5} \frac{1}{a^2}\left(\hat{n} \times (\hat{\beta} \times \hat{a})\right)^2
\]

The simplest way is to take \(\beta\) along \(z\)-axis and \(\hat{a}\) in the \(x-z\) plane and then do all integrals

\[
\hat{\beta} = (0, 0, \beta)
\]

\[
\hat{a} = (a_\perp, 0, a_\parallel)
\]

\[
\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
\]
Covariant Form of Total Power pg. 1

Then work out using \( \bar{\alpha} = \bar{\alpha}_x + \bar{\alpha}_y \)

\[ \int (\hat{n} \cdot \hat{\beta}) \times \bar{\alpha} \, d^2 \ell = \ldots \ldots \]

Plug into Eq \( \theta \), do all integrals. This gives the Lienard – Wiechert result, Eq. (1).

Covariant Form of Lienard – Wiechert Result/Larmour

This week homework discussed

\[ A^\mu = \frac{d^2 x^\mu}{d\tau^2} = \text{proper acceleration} \]

Can show \( U_\mu A^\mu = 0 \): Since \( U_\mu = U_\mu^\mu + 8U_\mu \)

with \( 8U_\mu = A_\mu \delta \tau \), then since \( U_\mu U^\mu = -c^2 \) is constant in time:

\[ U_\mu U^\mu = (U_\mu^\mu + 8U_\mu)(U^\mu + 8U^\mu) = -c^2 \]

\[ = U_\mu^\mu U^\mu + 2U_\mu^\mu A^\mu \delta \tau = -c^2 \]

So find

\[ U_\mu A^\mu = 0 \]
Covariant Form of Lienard-Wiechert/Larmour Result Pg. 2

Thus in LRF of particle (LRF = Local Rest Frame)

\[ A^\mu = \begin{pmatrix} 0 \\ \alpha_{\parallel} \\ \alpha_\perp \end{pmatrix} \quad A^\mu A_\mu = \alpha_{\parallel}^2 + \alpha_\perp^2 \]

Show that

\[ \alpha_{\parallel} = \frac{\alpha_{\parallel}}{\gamma^3} \]
\[ \alpha_\perp = \frac{\alpha_\perp}{\gamma^2} \]

So

\[ \gamma^6 \left[ \alpha_{\parallel} + \frac{\alpha_\perp^2}{\gamma^2} \right] = \alpha_{\parallel}^2 + \alpha_\perp^2 = A^\mu A_\mu \]

So we see that the Lienard-Wiechert result can be written

\[ P(T) = \frac{e^2}{4\pi} \frac{2}{3} \frac{A^\mu A_\mu}{c^3} \]